Geometric bias in eigenspace perturbation under random heterogeneous noise

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Abstract

Spectral methods rely fundamentally on the stability of principal eigenspaces under random perturbations. Classically, this stability is quantified by the Davis-Kahan and Wedin theorems, which bound the eigenspace error using the operator norm of the noise and the relevant spectral gaps. While these worst-case bounds are sharp for arbitrary deterministic perturbations, they can be wasteful in the low-rank signal-plus-random-noise setting, as they fail to capture the fine-grained interaction between the signal geometry and the noise distribution. In this paper, we study the spectral perturbation of signal-plus-noise matrices corrupted by sparse, random noise with an arbitrary, inhomogeneous variance profile. We demonstrate that under heterogeneous noise variances, the empirical eigenvectors suffer a systematic, deterministic geometric bias that is entirely invisible to classical perturbation bounds. By leveraging the Quadratic Vector Equation (QVE) and establishing fine-grained isotropic local laws, we derive near-optimal, non-asymptotic perturbation bounds for the leading eigenspaces in the operator and $2 \to \infty$ norms. The bounds separate the usual signal-to-noise contribution, stochastic fluctuations, and structured geometric bias terms determined by the alignment between the signal eigenspaces and the row-wise variance profile.

1 Introduction

Spectral methods are a basic tool for extracting low-dimensional structure from high-dimensional noisy data. In many statistical and network models, the observed matrix consists of a low-rank signal corrupted by random noise with independent but non-identically distributed entries. Classical perturbation theory, including Weyl's inequality and the Davis-Kahan-Wedin $\sin \Theta$ theorem, bounds the error in empirical eigenspaces using the operator norm of the noise and the spectral gaps. However, when the signal is low-rank and the perturbation comes from random noise, these worst-case deterministic bounds are inherently conservative. While recent work has developed refined bounds that exploit stochastic structure (discussed further in Section 4), we identify a key feature of heterogeneous random noise: the variance profile creates a systematic geometric bias in the empirical eigenvectors. Even when signal directions are orthogonal in the standard inner product, they may not be orthogonal with respect to the variance-weighted inner product induced

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by the noise. This paper develops non-asymptotic eigenspace perturbation bounds that separate three distinct contributions: the standard signal-to-noise ratio, the random fluctuations controlled by isotropic local laws, and the deterministic bias induced by the variance profile.

Our model is

$$\widetilde{A} = A + E,$$

where $A \in \mathbb{R}^{n\times n}$ is a deterministic matrix with rank r, representing the signal, and E is the random noise. Denote the eigen-decomposition of A as $A = U\Lambda U^{\mathrm{T}} = \sum_{i=1}^{r} \lambda_i u_i u_i^{\mathrm{T}}$ where the non-zero eigenvalues of A are ordered algebraically:

$$\lambda_1 \geqslant \cdots \geqslant \lambda_r.$$

The ordered eigenvalues and their corresponding eigenvectors of $\widetilde{A}$ are denoted as $\widetilde{\lambda}_i, \widetilde{u}_i$ for $1 \leqslant i \leqslant n$.

Let

$$r_+ := \#\{j : \lambda_j > 0\}, \qquad r_- := r - r_+,$$

so that

$$\lambda_1 \geqslant \cdots \geqslant \lambda_{r_+} > 0 > \lambda_{r_+ +1} \geqslant \cdots \geqslant \lambda_r.$$

In the main text we focus on the right-edge outliers and the top-k ($1 \leqslant k \leqslant r_+$) positive signal eigenspace and its perturbed counterpart

$$U_k := (u_1, \ldots, u_k),$$
$$\widetilde{U}_k := (\widetilde{u}_1, \ldots, \widetilde{u}_k).$$

The left-edge (negative-spike) case is entirely analogous and follows by applying the same argument to $-A$ (see Remark 2.1). Denote $\Lambda_k = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$ and define $\widetilde{\Lambda}_k$ analogously. For any index set $J \subseteq [r]$, U_J denotes the submatrix of U with columns indexed by J.

We impose the following assumptions on random noise E:

ASSUMPTION 1.1 (Structural assumption on the noise)**.** *Let E be an $n \times n$ symmetric random matrix with independent centered entries (up to symmetry) and variance profile $\Sigma = (\sigma_{ij}^2)_{1\leqslant i,j\leqslant n}$ where $\mathbb{E}(E_{ij}^2) = \sigma_{ij}^2$. Assume there exists a parameter $K \geqslant 1$ such that the sub-Gaussian norm* of each entry satisfies:*

$$\|E_{ij}\|_{\psi_2} \leqslant K\sigma_{ij}.$$

Equivalently, we can write $E_{ij} = \sigma_{ij}\xi_{ij}$ where the random variables ξ_{ij} have mean 0, variance 1, and satisfy $\max_{i,j} \|\xi_{ij}\|_{\psi_2} \leqslant K$.

Let $\sigma_{\max} := \max_{i,j} \sigma_{ij} > 0$. We further assume that there exists a parameter $\beta \geqslant 1$ such that

$$\mathbb{E}|E_{ij}|^3 \leqslant \beta K \sigma_{\max}\, \sigma_{ij}^2, \qquad \mathbb{E}|E_{ij}|^4 \leqslant \beta^2 K^2 \sigma_{\max}^2\, \sigma_{ij}^2. \tag{1.1}$$

*The sub-Gaussian norm is of a random variable X is defined as $\|X\|_{\psi_2} := \inf\{t > 0 : \mathbb{E}(\exp(X^2/t^2)) \leqslant 2\}$. In particular, $\mathbb{P}(|X| \geqslant t) \leqslant 2\exp(-t^2/\|X\|_{\psi_2}^2)$. See [78] for details.

REMARK 1.1 (The role of β). *The parameter β is introduced to prevent artificial logarithmic losses in sparse regimes by decoupling the moment controls from the sub-Gaussian norm K. Relying solely on K, a standard truncation argument yields:*

$$\mathbb{E}|E_{ij}|^3 \leqslant C\sqrt{\log(eK)}K\sigma_{\max}\sigma_{ij}^2, \qquad \mathbb{E}|E_{ij}|^4 \leqslant C\log(eK)K^2\sigma_{\max}^2\sigma_{ij}^2,$$

which corresponds to taking $\beta \asymp \sqrt{\log(eK)}$. For a sparse centered Bernoulli(p) variable ξ, $K \lesssim p^{-1/2}$. This crude bound introduces a penalty of $\beta \asymp \sqrt{\log(1/p)}$. However, direct calculation shows $\mathbb{E}|\xi|^3, \mathbb{E}|\xi|^4 \leqslant \sigma^2$. The condition (1.1) *thus holds with $\beta = O(1)$, avoiding the unnecessary logarithmic loss.*

Although the parameter β in (1.1) *is allowed to depend on n, it should be understood as an effective moment parameter, chosen as small as possible. Under the sub-Gaussian assumption alone, the standard truncation argument shows that one may always take $\beta \lesssim \sqrt{\log(eK)}$.*

The spectral behavior of the noise matrix E is governed by the maximum row-wise variance $R_{\max} := \max_i \sum_j \sigma_{ij}^2$ and the maximum entry-wise variance $\sigma_{\max}^2 := \max_{i,j} \sigma_{ij}^2$. To capture the heterogeneity of the noise profile, we also define the *variance oscillation* parameter:

$$\operatorname{osc}(\mathcal{R}) = R_{\max} - R_{\min},$$

where $R_{\min} := \min_i \sum_j \sigma_{ij}^2$. While $R_{\max}$ and $\sigma_{\max}^2$ control the standard operator norm bounds, $\operatorname{osc}(\mathcal{R})$ specifically quantifies the geometric bias on the eigenvectors. For homogeneous or regular networks where $\operatorname{osc}(\mathcal{R}) \approx 0$, this deterministic bias vanishes.

Let us denote $R_i := \sum_{j=1}^n \sigma_{ij}^2$. Define the diagonal matrix

$$\mathcal{R} := \operatorname{diag}(R_1, \ldots, R_n)$$

and the *geometric bias matrix*

$$\mathcal{V} := U^{\mathrm{T}}\mathcal{R}U = (u_i^{\mathrm{T}}\mathcal{R}u_j)_{1\leqslant i,j\leqslant r}. \tag{1.2}$$

The diagonal term $\mathcal{V}_{ii} = u_i^{\mathrm{T}}\mathcal{R}u_i = \sum_k u_i(k)^2 R_k$ records the effective variance seen by u_i. The cross term $\mathcal{V}_{ij} = u_i^{\mathrm{T}}\mathcal{R}u_j = \sum_k u_i(k)u_j(k)R_k$ captures the geometric bias. Even u_i, u_j are orthogonal, they are generally not orthogonal with respect to the inner product induced by the heterogeneous noise weights $\mathcal{R}$.

ASSUMPTION 1.2 (General sub-Gaussian regime). *Fix $D > 0$, we assume*

$$\sqrt{R_{\max}} \geqslant C_0(D+8)K\sigma_{\max}\log n \quad \textit{and} \quad \lambda_1 \leqslant R_{\max}^3$$

for a large absolute constant $C_0 > 0$.

For specific noise distributions, we can weaken the first condition in Assumption 1.2:

ASSUMPTION 1.3 (Bounded sparse-entry regime). *Fix $D > 0$. Suppose that there exist absolute constants $c_0, \beta_0 > 0$ such that*

$$\max_{i,j} |E_{ij}| \leqslant c_0 K\sigma_{\max} \quad \textit{almost surely} \qquad \textit{and} \qquad \beta \leqslant \beta_0.$$

In this regime, we assume only

$$\sqrt{R_{\max}} \geqslant C_0(D+8)K\sigma_{\max}\sqrt{\log n} \quad \textit{and} \quad \lambda_1 \leqslant R_{\max}^3$$

for a large absolute constant $C_0 > 0$.

REMARK 1.2 (On the auxiliary Assumptions 1.2 and 1.3)**.** *The first condition in both assumptions prevents any single maximal entry from dominating its row sum. Under this condition, our estimate (see Lemma 5.4) yields* $\|E\| \leqslant 3\sqrt{R_{\max}}$ *with high probability, thereby guaranteeing a macroscopic signal-to-noise separation (*$\lambda_k \geqslant 3\|E\|$*) for any supercritical spike* $\lambda_k \geqslant 9\sqrt{R_{\max}}$*.*

For Assumption 1.2, the condition $\sqrt{R_{\max}} \geqslant C_0 K \sigma_{\max} \log n$ *is established using the sharp matrix concentration bounds of Bandeira and van Handel [14] combined with a standard truncation argument (see Lemma 5.4). By contrast, Assumption 1.3 relaxes this sparsity requirement to* $\sqrt{R_{\max}} \geqslant C_0 K \sigma_{\max} \sqrt{\log n}$*. This improved rate follows directly from [14, Corollary 3.2] and Talagrand's concentration inequality. The weaker condition* $\sqrt{R_{\max}} \gtrsim K \sigma_{\max} \sqrt{\log n}$ *in Assumption 1.3 is also available for real symmetric sub-Gaussian entries by the recent sharp matrix concentration results [28, Theorem 1.6].*

Both assumptions naturally capture sparse regimes. For example, in a stochastic block model with edge probabilities of order p*, one has* $R_{\max} \asymp np$*,* $\sigma_{\max} \asymp \sqrt{p}$*, and for centered Bernoulli noise,* $K \lesssim p^{-1/2}$*. Under Assumption 1.2, the condition becomes* $p \gtrsim \log^2 n/n$*. Under the bounded sparse-entry regime in Assumption 1.3, it improves to* $p \gtrsim \log n/n$*.*

The second condition $\lambda_1 \leqslant R_{\max}^3$ *excludes ultra-strong signals and is imposed simply for technical convenience. When a population spike is massive, i.e.,* $\lambda_k \geqslant R_{\max}^3 \asymp \|E\|^6$*, its empirical counterpart* $\widetilde{\lambda}_k$ *exhibits negligible fluctuations, and the Green's function* $G(z) = (zI - E)^{-1}$ *near* $z \approx \widetilde{\lambda}_k$ *is dominated by* $z^{-1}I$*. Eigenvector perturbations in this regime follow directly from Neumann series expansions. Since this requires different, simpler arguments and is not our main focus, we exclude it; see [83] for a treatment without this upper bound.*

For technical convenience, we assume the polynomial regime

$$R_{\max} \leqslant n^{100} \qquad \text{and} \qquad \sigma_{\max}^{-1} \leqslant n^{100}.$$

The exponent 100 is arbitrary and can be replaced by any sufficiently large fixed constant. This condition is an extremely mild requirement that covers essentially all statistical models of interest. For example, in random graph models with bounded edge weights, we have $R_{\max} \leqslant n$. In heavy-tailed matrix models, entries are typically truncated at level n^c and $R_{\max} \leqslant n^{2c+1}$.

Define the *spectral gap*

$$\delta_k := \begin{cases} \lambda_k - \lambda_{k+1}, & \text{if } k < r_+; \\ \lambda_k - 3\sqrt{R_{\max}}, & \text{if } k = r_+. \end{cases}$$

With the convention that $\delta_0 := +\infty$.

We first state two simplified consequences of our main theorem. These statements are intended to display the dominant scales without the full block notation. Theorem 1 concerns an individual outlier eigenvalue and eigenvector, while Theorem 2 concerns the top-k positive eigenspace. The precise non-asymptotic result, Theorem 3 in Section 2, keeps the structured variance-profile couplings between different signal eigenspaces.

We use the standard principal angle notation. For two matrices $U, V \in \mathbb{R}^{n\times k}$ with orthonormal columns, the principal angles $\theta_1, \ldots, \theta_k \in [0, \pi/2]$ are defined by $\sigma_i(U^{\mathrm{T}}V) = \cos\theta_i$. We write

$$\sin\angle(U, V) = \operatorname{diag}(\sin\theta_1, \ldots, \sin\theta_k),$$

so that

$$\|\sin\angle(U,V)\| = \|(I-UU^{\mathrm{T}})V\|.$$

When discussing subspaces, we identify U and V with their column spaces. For $k=1$, this reduces to the usual sine of the angle between two vectors. See [83] for further discussion.

THEOREM 1 (Individual eigenvalue and eigenvector perturbations: simplified version)**.** *Fix $D>0$. Under Assumptions 1.1 and 1.2, suppose that λ_k with $k\in[r_+]$ is a simple eigenvalue satisfying $\lambda_k\geqslant 9\sqrt{R_{\max}}$ and*

$$\delta_k^* := \min\{\delta_{k-1},\delta_k\} \gtrsim \sqrt{k}\left(r\beta K\sigma_{\max}(\log n)^2 + \frac{\mathrm{osc}(\mathcal{R})}{\lambda_k}\right),$$

then the following holds with probability at least $1-n^{-D}$,

$$\left|\widetilde{\lambda}_k - \lambda_k - \frac{\mathcal{V}_{kk}}{\lambda_k}\right| \lesssim r\beta K\sigma_{\max}(\log n)^2 + \frac{\mathrm{osc}(\mathcal{R})}{\lambda_k},$$

and

$$|\sin\angle(u_k,\widetilde{u}_k)| \lesssim \sqrt{k}\frac{\mathrm{osc}(\mathcal{R})}{\delta_k^*\lambda_k} + \sqrt{k}\frac{r\beta K\sigma_{\max}(\log n)^2}{\delta_k^*} + \frac{\sqrt{R_{\max}}}{\lambda_k},$$

$$\min_{\mathsf{s}\in\{\pm 1\}}\|\widetilde{u}_k - \mathsf{s}u_k\|_\infty \lesssim \sqrt{k}\|U\|_{2,\infty}\left(\frac{\mathrm{osc}(\mathcal{R})}{\delta_k^*\lambda_k} + \frac{R_{\max}}{\lambda_k^2} + \frac{r\beta K\sigma_{\max}(\log n)^2}{\delta_k^*}\right) + \sqrt{k}\frac{r\beta K\sigma_{\max}(\log n)^2}{\lambda_k}.$$

THEOREM 2 (Top-k eigenspace perturbations: simplified version)**.** *Fix $D>0$. Under Assumptions 1.1 and 1.2, suppose that λ_k with $k\in[r_+]$ satisfies $\lambda_k\geqslant 9\sqrt{R_{\max}}$ and*

$$\delta_k \gtrsim \sqrt{k}\left(r\beta K\sigma_{\max}(\log n)^2 + \frac{\mathrm{osc}(\mathcal{R})}{\lambda_k}\right),$$

then the following holds with probability at least $1-n^{-D}$:

(i) *For the operator norm bound,*

$$\|\sin\angle(U_k,\widetilde{U}_k)\| \lesssim \sqrt{k}\frac{\mathrm{osc}(\mathcal{R})}{\delta_k\lambda_k} + \sqrt{k}\frac{r\beta K\sigma_{\max}(\log n)^2}{\delta_k} + \frac{\sqrt{R_{\max}}}{\lambda_k}.$$

(ii) *For the $2\to\infty$ norm bound,*

$$\min_{O\in\mathbb{O}(k)}\|\widetilde{U}_k - U_kO\|_{2,\infty} \lesssim \sqrt{k}\|U\|_{2,\infty}\left(\frac{\mathrm{osc}(\mathcal{R})}{\delta_k\lambda_k} + \frac{r\beta K\sigma_{\max}(\log n)^2}{\delta_k} + \frac{R_{\max}}{\lambda_k^2}\right) + \sqrt{k}\frac{r\beta K\sigma_{\max}(\log n)^2}{\lambda_k}.$$

Under the bounded sparse-entry regime in Assumption 1.3, the same simplified bounds hold with $r\beta K\sigma_{\max}(\log n)^2$ replaced by $(c_0+\beta_0)rK\sigma_{\max}\log n$, up to absolute constants.

Roadmap. The paper is organized as follows. Section 2 presents our main perturbation theorem with structured geometric couplings and both operator-norm and rowwise error bounds. Section 3 explains the geometric bias mechanism through a rank-two example and the QVE expansion, including numerical experiments and an oracle debiasing procedure. Section 4 reviews related literature. The technical development is presented in Sections 5–9: we establish the QVE expansion that generates the variance-profile bias (Section 5), state the probabilistic inputs including the isotropic local law and outlier eigenvalue locations (Section 6), prove the main perturbation theorem using blockwise eigenvector equations (Section 7), and establish the local law and outlier eigenvalue results (Sections 8–9). Additional technical results appear in the appendices.

Notation. For a vector $v=(v_1,\ldots,v_n)^{\mathrm{T}}\in\mathbb{R}^n$, we let $\|v\|$ denote its Euclidean ℓ_2 norm and $\|v\|_\infty:=\max_i|v_i|$ denote its maximum absolute entry. The oscillation of a vector v is denoted by $\mathrm{osc}(v):=\max_i v_i-\min_i v_i$. For a matrix B, we define its off-diagonal part as $\mathrm{Off}(B):=B-\mathrm{diag}(B)$, where $\mathrm{diag}(B)$ is the diagonal matrix formed by the diagonal entries of B. We use various matrix norms throughout the paper. Let $\|B\|$ denote the operator norm, and let $\|B\|_F$ denote the Frobenius norm. The $2\to\infty$ norm of a matrix is defined as $\|B\|_{2,\infty}:=\max_i\|e_i^{\mathrm{T}}B\|$, which is the maximum Euclidean norm of the rows of B. The infinity operator norm is denoted by $\|B\|_{\infty\to\infty}:=\max_i\sum_j|B_{ij}|$. We let $\mathbb{O}(k)$ denote the set of $k\times k$ orthogonal matrices. For two non-negative sequences X_n and Y_n, we write $X_n\lesssim Y_n$ (or $Y_n\gtrsim X_n$) if there exists an absolute constant $c>0$ such that $X_n\leqslant cY_n$ for all sufficiently large n. We write $X_n\asymp Y_n$ if both $X_n\lesssim Y_n$ and $X_n\gtrsim Y_n$ hold. Throughout the paper, C,C' denote positive absolute constants whose exact values may change from line to line.

2 Main perturbation results

We introduce the detailed non-asymptotic perturbation results for the top-k eigenspaces. Throughout the main results, we fix a parameter $D>0$. Under the general sub-Gaussian regime (Assumption 1.2), we define

$$M_D^{\mathrm{gen}}:=C_{\mathrm{gen}}(D+6)^{3/2}rK\sigma_{\max}(\log n)^2,$$

where $C_{\mathrm{gen}}>0$ is a sufficiently large absolute constant. In the bounded sparse-entry regime (Assumption 1.3), we can improve M_D^{gen} to

$$M_D^{\mathrm{bd}}:=C_{\mathrm{bd}}(c_0+\beta_0)(D+6)r\beta K\sigma_{\max}\log n.$$

Here, $C_{\mathrm{bd}}>0$ is a sufficiently large absolute constant.

For simplicity, we denote

$$M\equiv M_D:=\begin{cases}M_D^{\mathrm{gen}}, & \text{under Assumption 1.2,}\\ M_D^{\mathrm{bd}}, & \text{under Assumption 1.3.}\end{cases}\tag{2.1}$$

Furthermore, for fixed $k \in [r_+]$, define index sets

$$\mathcal{J} := \{k+1, \ldots, r_+\}, \quad \mathcal{I} := [r] \backslash \mathcal{J},$$
$$\mathcal{N} := \{r_+ + 1, \ldots, r\}, \quad \mathcal{K} := [r] \backslash \mathcal{N}.$$

and denote

$$\mathcal{B}_k := \frac{10\sqrt{k}}{\delta_k \lambda_k} \left(\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| + 8 \frac{R_{\max}}{\lambda_k^2} \mathrm{osc}(\mathcal{R}) \right) + 4\sqrt{k} \frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} \tag{2.2}$$

where $\mathcal{V}_{\mathcal{J}\mathcal{I}} = U_{\mathcal{J}}^{\mathrm{T}} \mathcal{R} U_{\mathcal{I}}$ and $\mathcal{V}_{\mathcal{N}\mathcal{K}} = U_{\mathcal{N}}^{\mathrm{T}} \mathcal{R} U_{\mathcal{K}}$. If $k = r_+$ (thus $\mathcal{J} = \varnothing$), then $\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| = 0$. We also set

$$\rho_k \equiv \rho_{k,D} := 10 M_D + 15 \frac{\mathrm{osc}(\mathcal{R})}{\lambda_k}.$$

THEOREM 3 (Top-k eigenspace perturbations). *Fix $D > 0$. Under Assumption 1.1 and one of Assumptions 1.2 and 1.3, for any $k \in [r_+]$ such that $\lambda_k \geqslant 9\sqrt{R_{\max}} + 100\rho_{k,D}$ and*

$$\delta_k \geqslant \sqrt{k} \left(65 \rho_{k,D} + 20 \frac{\mathrm{osc}(\mathcal{R})}{\lambda_k} \right),$$

we have the following holds with probability at least $1 - n^{-D}$:

(i) *For the operator norm bound, we have*

$$\| \sin \angle (U_k, \widetilde{U}_k) \| \leqslant \mathcal{B}_k + 10\sqrt{k} \frac{M_D}{\delta_k} + 2 \frac{\|E\|}{\lambda_k}.$$

(ii) *For the $2 \to \infty$ norm bound, we first have*

$$\|\widetilde{U}_k - P_{U_k} \widetilde{U}_k\|_{2,\infty} \leqslant \|U\|_{2,\infty} \left(\mathcal{B}_k + 10\sqrt{k} \frac{M_D}{\delta_k} + 8\sqrt{k} \frac{R_{\max}}{\lambda_k^2} \right) + 15\sqrt{k} \frac{M_D}{\lambda_k}.$$

Consequently,

$$\begin{aligned} \min_{O \in \mathbb{O}(k)} \|\widetilde{U}_k - U_k O\|_{2,\infty} &\leqslant \|\widetilde{U}_k - P_{U_k} \widetilde{U}_k\|_{2,\infty} + \|U_k\|_{2,\infty} \| \sin \angle (U_k, \widetilde{U}_k) \|^2, \\ &\leqslant 4 \|U\|_{2,\infty} \left(\mathcal{B}_k + 20\sqrt{k} \frac{M_D}{\delta_k} + 30\sqrt{k} \frac{R_{\max}}{\lambda_k^2} \right) + 120\sqrt{k} \frac{M_D}{\lambda_k}. \end{aligned}$$

The detailed bounds in Theorem 3 retain a geometric bias term $\mathcal{B}_k$ defined in (2.2). In particular, it retains the quantities

$$\mathcal{V}_{\mathcal{J}\mathcal{I}} = U_{\mathcal{J}}^{\mathrm{T}} \mathcal{R}\, U_{\mathcal{I}}, \qquad \mathcal{V}_{\mathcal{N}\mathcal{K}} = U_{\mathcal{N}}^{\mathrm{T}} \mathcal{R}\, U_{\mathcal{K}},$$

which encode how the signal eigenspaces align with the row-variance profile $\mathcal{R} = \mathrm{diag}(R_i)$. These quantities vanish in the homogeneous case when $\mathcal{R} = cI$. More generally, by orthogonality,

$$\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| = \|U_{\mathcal{J}}^{\mathrm{T}} (\mathcal{R} - cI)\, U_{\mathcal{I}}\| \leqslant \frac{1}{2} \mathrm{osc}(\mathcal{R})$$

by choosing $c = \frac{1}{2}(R_{\max} + R_{\min})$. Likewise, $\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\| \leqslant \frac{1}{2} \mathrm{osc}(\mathcal{R})$. Therefore, the simplified theorems in the Introduction follow directly from Theorem 3. Indeed, the simplified dependence on $\mathrm{osc}(\mathcal{R})$ in Theorem 2 should therefore be viewed as a worst-case bound for these couplings $\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|, \|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|$.

REMARK 2.1 (Bounds for general eigenspaces)**.** *Theorem 3 is stated for the top positive eigenspace only to simplify notation. The proof extends directly to general eigenspaces $U_{l:s} = (u_l, \dots, u_s)$ for $1 \leqslant l < s \leqslant r$. We also provide a simple alternative approach using the bounds for top eigenspaces.*

For negative population spikes, we apply our framework to $-A$ and $-(A+E)$. In our ordering $\lambda_1 \geqslant \cdots \geqslant \lambda_{r_+} > 0 > \lambda_{r_+ +1} \geqslant \ldots \geqslant \lambda_r$, the eigenspaces corresponding to the most negative eigenvalues become the top eigenspaces. This symmetry yields the same perturbation guarantees for the bottom eigenspaces without additional analysis.

For intermediate blocks $U_{l:s}$ and $\widetilde{U}_{l:s}$ where $1 \leqslant l < s \leqslant r_+$. We use the decomposition $P_{U_{l:s}} = P_{U_s} - P_{U_{l-1}}$ to obtain

$$\| \sin \angle(U_{l:s}, \widetilde{U}_{l:s})\| \leqslant \| \sin \angle(U_{l-1}, \widetilde{U}_{l-1})\| + \| \sin \angle(U_s, \widetilde{U}_s)\|, \tag{2.3}$$

and

$$\min_O \|\widetilde{U}_{l:s} - U_{l:s}O\|_{2,\infty} \leqslant \|(I - P_{U_{l:s}})\widetilde{U}_{l:s}\|_{2,\infty} + \|U_{l:s}\|_{2,\infty}\| \sin \angle(U_{l:s}, \widetilde{U}_{l:s})\|^2, \tag{2.4}$$

where

$$\|(I - P_{U_{l:s}})\widetilde{U}_{l:s}\|_{2,\infty} \leqslant \|(I - P_{U_s})\widetilde{U}_s\|_{2,\infty} + \|U_{l-1}\|_{2,\infty}\| \sin \angle(U_{l-1}, \widetilde{U}_{l-1})\|.$$

Thus, the perturbation bounds for $U_{l:s}, \widetilde{U}_{l:s}$ follow by substituting the estimates for the top-s and top-$(l-1)$ eigenspaces. By symmetry, analogous decompositions hold for intermediate blocks of negative spikes. We prove (2.3) *and* (2.4) *in Appendix F.*

Finally, we formulate a perturbation result for the entire eigenspace $U = (u_1, \dots, u_r)$. Denote

$$\widetilde{U}_{\text{sig}} := (\widetilde{u}_1, \dots, \widetilde{u}_{r_+}, \widetilde{u}_{n-r_-+1}, \dots, \widetilde{u}_n).$$

Thus $\widetilde{U}_{\text{sig}}$ collects the empirical right-edge and left-edge outliers corresponding to the positive and negative population spikes. In this case, the relevant spectral gap is the distance from the signal spectrum to the bulk at 0, namely

$$\lambda_{\min} := \min_{1 \leqslant j \leqslant r} |\lambda_j|.$$

The principal angle bound is given directly by the classical Davis-Kahan/Wedin theorem, so we only record the $2 \to \infty$ perturbation estimate for the full signal eigenspace. Perturbation bounds for $U, \widetilde{U}_{\text{sig}}$ follow by combining the positive- and negative-spike versions above.

COROLLARY 1 (Full signal eigenspace)**.** *Fix $D > 0$. Under Assumption 1.1 and one of Assumptions 1.2 and 1.3, assume*

$$\lambda_{\min} \geqslant 9\sqrt{R_{\max}} + 165\sqrt{r}\rho_{k,D} + 20\sqrt{r}\frac{\text{osc}(\mathcal{R})}{\lambda_{\min}},$$

then, with probability at least $1 - n^{-D}$,

$$\min_{O \in \mathbb{O}(r)} \|\widetilde{U}_{\text{sig}} - UO\|_{2,\infty} \leqslant 42\sqrt{r}\|U\|_{2,\infty}\frac{\text{osc}(\mathcal{R})}{\lambda_{\min}^2} + 240\|U\|_{2,\infty}\sqrt{r}\frac{R_{\max}}{\lambda_{\min}^2} + 480\sqrt{r}\frac{M_D}{\lambda_{\min}}.$$

REMARK 2.2 (Extension to rectangular matrices)**.** *Our eigenspace perturbation results extend to singular subspace perturbation for rectangular matrices via the standard Hermitian dilation technique. See Appendix G for a brief discussion.*

REMARK 2.3 (Other perturbation metrics)**.** *Theorem 3 focuses on the operator-norm principal angle and the $2 \to \infty$ norm bounds for clarity. The proof framework is more general: it controls the block components of the empirical eigenspace along non-target signal directions and the null space of A. These estimates, combined with the techniques from [83], extend to all standard metrics (Frobenius norm, unitarily invariant norms, linear forms, max norm, and weighted rowwise norms).*

We focus on these two bounds because they reveal the key mechanisms most clearly. The operator-norm bound captures the projection of $\widetilde{U}_k$ outside U_k, including components along lower positive spikes, negative spikes, and the null space. The $2 \to \infty$ bound translates this decomposition into rowwise estimates, showing how the variance-profile bias interacts with the incoherence of the signal eigenvector.

3 Geometric bias, sharpness, and debiasing

3.1 Interpretation of the geometric bias

The present variance-profile model introduces a new deterministic geometric bias term

$$\mathcal{B}_k = \frac{10\sqrt{k}}{\delta_k \lambda_k}\left(\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| + 8\frac{R_{\max}}{\lambda_k^2}\operatorname{osc}(\mathcal{R})\right) + 4\sqrt{k}\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2},$$

which captures the interaction between the signal eigenspaces and the inhomogeneous row-variance profile $\mathcal{R} = \operatorname{diag}(R_i)$. In the homogeneous noise setting where $\mathcal{R} = cI$, the geometric bias term $\mathcal{B}_k$ vanishes, our current bounds in Theorem 3 match the two-term bounds studied in [83], up to logarithmic factors and rank factors.

The two quantities $\mathcal{V}_{\mathcal{J}\mathcal{I}}, \mathcal{V}_{\mathcal{N}\mathcal{K}}$ play different roles in the bias term. The term $\mathcal{V}_{\mathcal{J}\mathcal{I}}$ captures how the target top-k cluster interacts with the lower positive cluster $\{k+1, \dots, r_+\}$, appearing at the gap-dependent scale $(\delta_k \lambda_k)^{-1}$. By contrast, $\mathcal{V}_{\mathcal{N}\mathcal{K}}$ captures interactions between negative and nonnegative signal directions, but at the weaker scale λ_k^{-2}. In practice, the contribution from $\mathcal{V}_{\mathcal{N}\mathcal{K}}$ is often negligible. For instance, when all spikes are nonnegative, we have $\mathcal{N} = \varnothing$ and this term vanishes. Even when negative spikes exist, the term involving $\mathcal{V}_{\mathcal{N}\mathcal{K}}$ is usually dominated by $\mathcal{V}_{\mathcal{J}\mathcal{I}}$ as it has a stronger suppression factor λ_k^{-2}.

We believe that the leading geometric bias terms

$$\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\delta_k \lambda_k}, \quad \frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2}$$

are intrinsic to the variance-profile model.

A rank-two model illustration. To explain where these terms come from, consider a simple rank-2 model $A = \lambda_1 u_1 u_1^{\mathrm{T}} + \lambda_2 u_2 u_2^{\mathrm{T}}$ with $\lambda_1 > \lambda_2 > 0$ and the leading empirical eigenvector

$\widetilde{u}_1$. Let Q be the orthogonal projection onto the null space of A. From the decomposition $\widetilde{u}_1 = u_1 u_1^{\mathrm{T}} \widetilde{u}_1 + u_2 u_2^{\mathrm{T}} \widetilde{u}_1 + Q\widetilde{u}_1$, taking the ℓ_2 norm on both sides and rearranging terms, we have

$$\sin^2 \angle(u_1, \widetilde{u}_1) = (u_2^{\mathrm{T}} \widetilde{u}_1)^2 + \|Q\widetilde{u}_1\|^2.$$

The term $u_2^{\mathrm{T}} \widetilde{u}_1$ captures the interaction between signal directions. In our proofs, this quantity is controlled by projecting the eigenvector equation $\widetilde{u}_1 = G(\widetilde{\lambda}_1) A\widetilde{u}_1$ onto u_2. To precisely estimate this term, from the eigenvector equation $(A + E)\widetilde{u}_1 = \widetilde{\lambda}_1 \widetilde{u}_1$, we solve for

$$\widetilde{u}_1 = (\widetilde{\lambda}_1 - E)^{-1} A\widetilde{u}_1 = G(\widetilde{\lambda}_1) A\widetilde{u}_1.$$

A key technical input in our proof is to show that the random resolvent G can be precisely approximated by its QVE approximation, a deterministic matrix Φ (defined (5.1)). This Φ isolates the deterministic contribution of the variance profile. Indeed, from

$$u_2^{\mathrm{T}} \widetilde{u}_1 \approx u_2^{\mathrm{T}} \Phi(\widetilde{\lambda}_1) A\widetilde{u}_1 = \lambda_1 (u_2^{\mathrm{T}} \Phi(\widetilde{\lambda}_1) u_1) u_1^{\mathrm{T}} \widetilde{u}_1 + \lambda_2 (u_2^{\mathrm{T}} \Phi(\widetilde{\lambda}_1) u_2) u_2^{\mathrm{T}} \widetilde{u}_1,$$

plugging in the expansion of $\Phi \approx z^{-1} I + z^{-3} \mathcal{R}$ in (5.3), the leading off-diagonal term is

$$u_2^{\mathrm{T}} \Phi(\widetilde{\lambda}_1) u_1 \approx \frac{u_2^{\mathrm{T}} \mathcal{R} u_1}{\widetilde{\lambda}_1^3}.$$

Substituting this back and approximating $\widetilde{\lambda}_1 \approx \lambda_1$ and $u_1^{\mathrm{T}} \widetilde{u}_1 \approx 1$, we obtain

$$u_2^{\mathrm{T}} \widetilde{u}_1 \approx \frac{1}{\lambda_1^2} (u_2^{\mathrm{T}} \mathcal{R} u_1) + \frac{\lambda_2}{\lambda_1} u_2^{\mathrm{T}} \widetilde{u}_1.$$

Rearranging this equation to solve $u_2^{\mathrm{T}} \widetilde{u}_1$ yields

$$|u_2^{\mathrm{T}} \widetilde{u}_1| \approx \frac{|u_2^{\mathrm{T}} \mathcal{R} u_1|}{\delta_1 \lambda_1},$$

which mirrors the $\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\delta_k \lambda_k}$ term in our theorem. If instead $\lambda_1 > 0 > \lambda_2$, the same calculation reveals

$$|u_2^{\mathrm{T}} \widetilde{u}_1| \approx \frac{|u_2^{\mathrm{T}} \mathcal{R} u_1|}{\lambda_1 (\lambda_1 - \lambda_2)} = \frac{|u_2^{\mathrm{T}} \mathcal{R} u_1|}{\lambda_1 (\lambda_1 + |\lambda_2|)}.$$

We bound $\frac{|u_2^{\mathrm{T}} \mathcal{R} u_1|}{\lambda_1 (\lambda_1 + |\lambda_2|)} \leqslant \frac{|u_2^{\mathrm{T}} \mathcal{R} u_1|}{\lambda_1^2}$ in our perturbation results for simplicity.

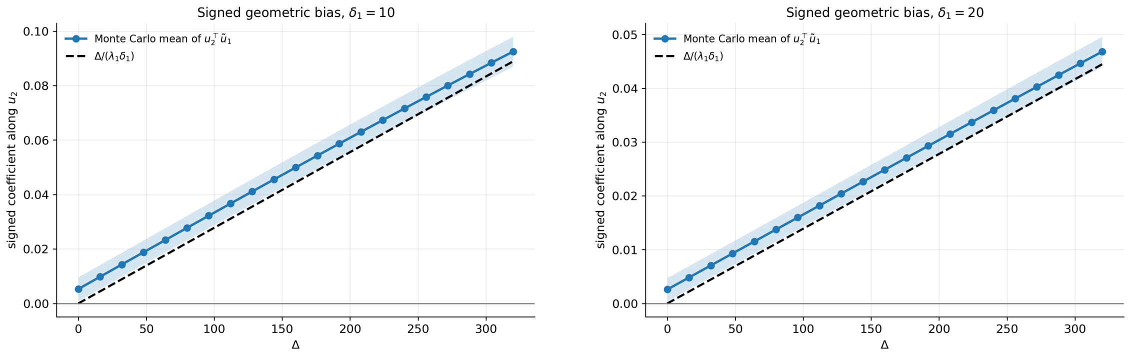
The preceding calculation explains the leading bias terms $\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\delta_k \lambda_k}, \frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2}$. By contrast, the higher-order term in $\mathcal{B}_k$ involving

$$\frac{R_{\max}}{\delta_k \lambda_k^3} \operatorname{osc}(\mathcal{R})$$

comes from bounding the diagonal remainder $\varepsilon(z)$ in the second-order expansion of $\Phi(z)$ in (5.1):

$$\Phi(z) = \frac{1}{z} I_n + \frac{1}{z^3} \mathcal{R} + \varepsilon(z).$$

We control this remainder using a uniform oscillation estimate, which may not be sharp. A higher-order expansion of $\Phi(z)$ would reveal additional structured variance-profile terms beyond $\mathcal{R}$ and replace the uniform $\operatorname{osc}(\mathcal{R})$ bound with finer geometric couplings, yielding a sharper higher-order term. However, isolating these higher-order structures requires substantially more delicate analysis to control the deterministic remainders and extract the finer stochastic fluctuations. This is beyond the scope of the current paper.



(a) $\delta_1 = 10$. (b) $\delta_1 = 20$.

Figure 1: Geometric bias under varying spectral gaps. Setup: dimension $n = 1200$, baseline noise $\rho = 400$, top spike $\lambda_1 = 360$, heterogeneity parameter $\Delta \in \{0, 16, 32, \dots, 320\}$, with 300 Monte Carlo samples. Each panel shows the empirical alignment $\mathbb{E}[u_2^{\mathrm{T}} \widetilde{u}_1]$ versus Δ, compared with the theoretical prediction $\Delta/(\lambda_1 \delta_1)$.

3.2 Sharpness and numerical experiments

Before presenting numerical experiments, we briefly discuss the sharpness of our bounds. Our bounds contain three terms, compared to two terms in the homogeneous noise setting of [83]. In [83] for i.i.d. Gaussian noise, eigenspace perturbation is governed by two terms: a stochastic fluctuation term ($\propto \sqrt{r}/\delta_k$) and a signal-to-noise term ($\propto \|E\|/\lambda_k$). As shown in [83, Section 4.1], these terms are near-optimal.

Under heterogeneous variance profiles, in Theorem 3, a third term emerges: the geometric bias term $\mathcal{B}_k$. While the first two terms remain essentially unchanged (up to logarithmic factors and a $\sqrt{r}$ factor), this new term captures the deterministic bias induced by variance heterogeneity. Since the first two terms were already validated in [83], our simulations focus on the third term. We design experiments to isolate $\mathcal{B}_k$ and confirm that it is not an artifact of loose bounds, but a visible deterministic effect in heterogeneous regimes.

Numerical simulations. We consider a rank-2 model $A = \lambda_1 u_1 u_1^{\mathrm{T}} + \lambda_2 u_2 u_2^{\mathrm{T}}$ with spikes $\lambda_1 > \lambda_2 > 0$ and eigenvectors $u_1 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $u_2 = \frac{1}{\sqrt{2}}(e_1 - e_2)$. The noise variance profile has row sums $\mathcal{R}_\Delta = \mathrm{diag}(\rho + \Delta, \rho - \Delta, \rho, \dots, \rho)$, where $\rho > 0$ is the baseline variance and $0 \leqslant \Delta \leqslant 0.8\rho$ is the heterogeneity parameter. This gives the geometric coupling $\mathcal{V}_{12} = u_2^{\mathrm{T}} \mathcal{R}_\Delta u_1 = \Delta$.

To show that geometric bias is deterministic rather than random, we compute the Monte Carlo average of $u_2^{\mathrm{T}} \widetilde{u}_1$, where in each trial we align the empirical eigenvector so that $u_1^{\mathrm{T}} \widetilde{u}_1 \geqslant 0$. Since the noise is zero-mean and isotropic, averaging removes random fluctuations while preserving the deterministic bias. Figure 1 compares these empirical averages with our theoretical prediction $\frac{\Delta}{\delta_1 \lambda_1}$ for two different spectral gaps $\delta_1 = \lambda_1 - \lambda_2$.

3.3 Oracle de-biasing

The goal of this subsection is modest. We show, at the oracle level, that the leading $\mathcal{V}_{\mathcal{J}\mathcal{I}}$-bias in $\mathcal{B}_k$ can be removed. As discussed earlier, $\mathcal{V}_{\mathcal{J}\mathcal{I}}$ causes the main deviation from previous homogeneous bounds. When $k = r_+$, $\mathcal{J} = \varnothing$ and this bias vanishes. For intermediate eigenspaces ($k < r_+$), correction may help when the deterministic bias dominates the stochastic fluctuations, i.e.,

$$\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\lambda_k} \gg M.$$

For $s \in [k]$, consider the block matrices $U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{I}}$ and $U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{J}}$. Define

$$\mathcal{T}_s := (I - U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{J}} \Lambda_{\mathcal{J}})^{-1} U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{I}} \Lambda_{\mathcal{I}}. \tag{3.1}$$

The next result guarantees that $\mathcal{T}_s$ is well-defined on the event considered in Theorem 3. Its proof is provided in Appendix H.1.

LEMMA 3.1. *Assume $\mathcal{J} = \{k+1, \ldots, r_+\} \neq \varnothing$. Under the assumptions and with the same probability as in Theorem 3, for every $s \in [k]$,*

$$\left\| \left(I - U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{J}} \Lambda_{\mathcal{J}} \right)^{-1} \right\| \leqslant 60 \frac{\widetilde{\lambda}_s}{\delta_k}.$$

Define the oracle corrected vectors

$$w_s^{\mathrm{orc}} := P_U \widetilde{u}_s - U_{\mathcal{J}} \mathcal{T}_s U_{\mathcal{I}}^{\mathrm{T}} \widetilde{u}_s, \qquad s = 1, \ldots, k.$$

Let

$$W_k^{\mathrm{orc}} := (w_1^{\mathrm{orc}}, \ldots, w_k^{\mathrm{orc}}), \qquad U_k^{\mathrm{orc}} := \mathrm{orth}(W_k^{\mathrm{orc}}).$$

We denote by $\mathrm{orth}(W)$ the orthonormalization of the matrix W, which consists of orthonormal columns spanning the same subspace as W.

Define the oracle error term

$$\mathcal{E}_k^{\mathrm{orc}} := 4\sqrt{k} \frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 24\sqrt{k} \frac{R_{\max}}{\lambda_k^4} \mathrm{osc}(\mathcal{R}) + 124\sqrt{k} \frac{M_D}{\delta_k}.$$

PROPOSITION 3.1 (Oracle blockwise de-biasing)**.** *Assume that the conditions of Theorem 3 hold. The following holds with the same probability as Theorem 3:*

$$\|\sin \angle(U_k^{\mathrm{orc}}, U_k)\| \leqslant 2\mathcal{E}_k^{\mathrm{orc}},$$
$$\min_{O \in \mathbb{O}(k)} \|U_k^{\mathrm{orc}} - U_k O\|_{2,\infty} \leqslant 3\|U\|_{2,\infty} \mathcal{E}_k^{\mathrm{orc}}.$$

The proof of Proposition 3.1 is deferred to Appendix H.2. The key point is that the oracle correction subtracts the deterministic contribution from the $\mathcal{J}$-block predicted by the QVE. This removes the leading $\mathcal{V}_{\mathcal{J}\mathcal{I}}$-term in $\mathcal{B}_k$. Additionally, the projection P_U removes the null-space component, which explains why the signal-noise term $\|E\|/\lambda_k$ in Theorem 3 is absent from Proposition 3.1.

The oracle estimator uses the unknown population eigenspace. A natural plug-in version replaces the population blocks by empirical outlier blocks. We provide a plug-in de-biasing implementation and explain its limitations in the Appendix I. Developing optimal data-driven bias correction methods is left for future work.

4 Related literature

This section highlights perturbation results relevant to heterogeneous random noise. A broader review of matrix perturbation, rowwise eigenspace analysis, and statistical applications can be found in [83]. Here we focus on work most closely related to variance heterogeneity and random matrix perturbation with non-identically distributed noise.

Perturbation bounds beyond worst-case theory. Classical perturbation results, including Weyl's inequality [86] and the Davis-Kahan-Wedin $\sin\Theta$ theorem [47, 85], control eigenvalue and eigenspace errors through the operator norm of the perturbation and the relevant spectral separation. While sharp for arbitrary deterministic perturbations, these bounds can be conservative for structured random perturbations. Modern refinements include population-gap variants, Schatten and unitarily invariant norm bounds, perturbation projection error bounds, and other deterministic or stochastic improvements; see, for example, [91, 82, 34, 67, 93, 84, 7, 19, 95, 80, 72, 24].

Extensive work has also addressed entrywise and rowwise perturbation bounds. Representative works include [51, 2, 1, 32, 37, 42, 64, 96, 24, 89, 90, 88]. Perturbation of linear and bilinear forms of eigenvectors and singular vectors has been studied in [63, 87, 49, 65, 44, 3, 4]. These results play an important role in high-dimensional statistics, including matrix completion, spectral clustering, ranking, community detection, submatrix localization, and Gaussian mixture models; see, for example, [36, 35, 61, 8, 9, 33, 43, 69, 79, 81, 66, 13, 62, 30, 31, 68, 48, 52, 56, 70, 29, 46]. For a statistical survey of spectral methods and perturbation bounds, see [41].

In the homogeneous Gaussian-noise setting, [73, 83] obtained sharp singular-subspace perturbation estimates showing that the error is governed by a signal-to-noise term and a stochastic signal-space fluctuation term. Recent work of Tran and Vu [76, 77] develops combinatorial contour-expansion and relative-norm perturbation bounds that exploit structural interaction between the signal and the perturbation. Our work shares the goal of moving beyond worst-case perturbation theory, but through a different mechanism: we use variance-profile local laws and QVE expansions to identify a deterministic geometric bias induced by heterogeneous noise.

Heterogeneous noise in statistical spectral problems. There is a substantial statistical literature on spectral estimation under heteroskedastic or heterogeneous noise. Zhang, Cai and Wu [92] introduced HeteroPCA, based on diagonal deletion and imputation, to correct the bias caused by heterogeneous diagonal noise in spiked covariance models. Yan, Chen and Fan [89] developed inference for heteroskedastic PCA with missing data, obtaining distributional guarantees for the estimated principal subspace and entrywise inference for the spiked covariance matrix. Related work on high-dimensional heteroskedastic PCA and weighted PCA includes [58, 59, 60]. Zhang and Mondelli [94] recently studied rank-one matrix denoising with doubly heteroscedastic noise and derived asymptotic fundamental limits and optimal spectral methods.

These works are close in motivation but differ from ours in both model and mechanism. Much of this literature concerns sample covariance, missing-data, or rectangular denoising models, where the main issue is often diagonal-noise bias correction, whitening, optimal shrinkage, or statistical inference. By contrast, we study a symmetric additive signal-plus-noise model with arbitrary entrywise variance profile and possible sparsity. The bias identified here is not merely a diagonal variance bias; it is a geometric bias governed by $\mathcal{V} = U^{\mathrm{T}}\mathcal{R}U$, which records how the row-wise variance profile changes the geometry of the signal eigenspaces.

We also cite related work on rowwise, entrywise, and inferential perturbation for completeness, including [51, 37, 2, 1, 32, 64, 24, 89, 90, 4, 3, 44, 40]; these works concern fine-grained perturbation and distributional inference in related spectral models, but they do not address the variance-profile geometric bias studied here.

Related literature in random matrix theory. Our work is also connected to the random matrix literature on finite-rank deformations and outlier eigenvectors. The BBP transition was introduced by Baik, Ben Arous and Péché [11], with subsequent developments for spiked covariance models, matrix denoising, and finite-rank deformations in [12, 75, 10, 21, 22, 20, 27, 38, 39, 16, 17, 18, 23, 50, 71]. The variance-profile setting is closely related to the theory of general Wigner-type matrices and the quadratic vector equation. Ajanki, Erdős and Krüger [6] developed the QVE/Dyson-equation framework and proved local laws and universality for matrices with general variance profiles. Our use of the QVE is more elementary and takes place in the large-z outlier domain, where the solution is stable and admits a direct expansion.

Recent asymptotic work has begun to study outliers, detection, and inference limits in spiked models with variance-profile, inhomogeneous, or structured noise. Bhattacharya, Chakrabarty and Hazra [26] analyze outlier eigenvalues and eigenvectors of generalized Wigner matrices with finite-rank deterministic perturbations. Guionnet, Ko, Krzakala and Zdeborová [55] study low-rank matrix estimation with inhomogeneous output channels and detection thresholds. Bao, Cheong, Lee and Li [15] study signal detection from spiked noise via asymmetrization. We also mention the physics work of Ferreira and Metz [53] on the BBP transition in an inhomogeneous rank-one spiked Wigner model. These results are complementary to ours: they are mainly asymptotic and focus on limiting distributions, information limits, detection thresholds, or BBP transitions, whereas our results are finite-sample perturbation bounds for deterministic low-rank signals under sparse independent variance-profile noise.

5 Deterministic QVE estimates and preliminary bounds

Consider the resolvent (Green function) of E:

$$G(z) = (zI - E)^{-1}, \qquad z \in \mathbb{C}^+.$$

For $z \in \mathbb{C}^+$, define a deterministic diagonal matrix

$$\Phi \equiv \Phi(z) = \operatorname{diag}(\phi_1(z), \ldots, \phi_n(z))$$

where $\phi(z) = (\phi_1(z), \ldots, \phi_n(z))$ is the solution to the Quadratic Vector Equations (QVE) given by:

$$\frac{1}{\phi_i(z)} = z - \sum_{j=1}^{n} \sigma_{ij}^2 \phi_j(z), \qquad i \in [n], \tag{5.1}$$

with $\operatorname{Im} \phi_i(z) < 0$ for $z \in \mathbb{C}^+$. Each $\phi_i(z)$ is analytic on $\mathbb{C}^+$. It is shown in [6, Theorem 2.1] that such a solution exists, is unique, and extends continuously to the real axis outside the spectrum (so in particular $\phi_i(z) > 0$ for z sufficiently large).

We collect deterministic estimates for the QVE solution $\Phi(z)$ and elementary bounds for resolvents $G(z)$. These estimates are used in two places: to identify the deterministic locations of outlier eigenvalues, and to extract the geometric bias from the expansion of $\Phi(z)$.

Recall that

$$R_i = \sum_{j=1}^{n} \sigma_{ij}^2, \qquad R_{\max} = \max_i R_i, \qquad \mathcal{R} = \operatorname{diag}(R_1, \dots, R_n).$$

The next result provides basic bounds on $\phi_i(z)$ and the second-order expansion of these ϕ_i's for large z. The proof is provided in Appendix A.1.

LEMMA 5.1. *Assume $z \in \mathbb{C}$ satisfying $|z| \geqslant \sqrt{6R_{\max}}$.*
(i) *For every $i \in [n]$, we have*

$$\frac{1}{2|z|} \leqslant |\phi_i(z)| \leqslant \frac{3}{2|z|}$$

and

$$\frac{3}{4}|z| \leqslant \Big|z - \sum_j \sigma_{ij}^2 \phi_j(z)\Big| \leqslant \frac{5}{4}|z|.$$

Furthermore, we have

$$\|\Phi'(z)\| = \max_i |\phi_i'(z)| \leqslant \frac{4}{|z|^2} \quad \textit{and} \quad \|\Phi''(z)\| = \max_i |\phi_i''(z)| \leqslant \frac{28}{|z|^3}.$$

(ii) *For $i \in [n]$, we have the second-order expansion:*

$$\phi_i(z) = \frac{1}{z} + \frac{R_i}{z^3} + \varepsilon_i(z) \quad \textit{with} \quad |\varepsilon_i(z)| \leqslant \frac{45}{8}\frac{R_{\max}^2}{|z|^5}. \tag{5.2}$$

In particular,

$$\Phi(z) = \frac{1}{z} I_n + \frac{1}{z^3}\mathcal{R} + \varepsilon(z), \tag{5.3}$$

where

$$\varepsilon(z) = \operatorname{diag}(\varepsilon_1(z), \cdots, \varepsilon_n(z)) \quad \textit{and} \quad \|\varepsilon(z)\| \leqslant \frac{45}{8}\frac{R_{\max}^2}{|z|^5}.$$

Furthermore, we have

$$\Phi'(z) = -\frac{1}{z^2} I_n - \frac{3}{z^4}\mathcal{R} + \varepsilon'(z), \tag{5.4}$$

where

$$\varepsilon'(z) = \operatorname{diag}(\varepsilon_1'(z), \cdots, \varepsilon_n'(z)) \quad \textit{and} \quad \|\varepsilon'(z)\| \leqslant \frac{243}{5}\frac{R_{\max}^2}{|z|^6}.$$

The entrywise bound in Lemma 5.1 is not sufficient for the eigenspace perturbation theorem. For the error term $\varepsilon(z) = \operatorname{diag}(\varepsilon_1(z), \dots, \varepsilon_n(z))$ in the second-order expansion of $\Phi(z)$ in (5.2), we control its *oscillation*. For $z \in \mathbb{C}$ satisfying $|z| \geqslant \sqrt{6R_{\max}}$, denote

$$\operatorname{osc}(\varepsilon(z)) := \max_{i,j} |\varepsilon_i(z) - \varepsilon_j(z)|.$$

If z is a real number, then $\operatorname{osc}(\varepsilon(z)) = \max_i \varepsilon_i(z) - \min_i \varepsilon_i(z)$. Recall that

$$\operatorname{osc}(\mathcal{R}) = R_{\max} - R_{\min}.$$

The bound on $\operatorname{osc}(\varepsilon(z))$ is given below; its proof is deferred to Appendix A.2.

LEMMA 5.2. *For any complex number z with $|z| \geqslant \sqrt{6R_{\max}}$, we have*

$$\operatorname{osc}(\varepsilon(z)) \leqslant \frac{15R_{\max}}{|z|^5}\operatorname{osc}(\mathcal{R}).$$

The same QVE expansion also determines the typical locations around which the outlier eigenvalues concentrate. We prove the next result in Appendix A.3.

PROPOSITION 5.1 (Deterministic outlier location)**.** *Let $j \leqslant r_+$ and assume $\lambda_j \geqslant 6\sqrt{R_{\max}}$. Define*

$$\widehat{\alpha}_j(z) := 1 - \lambda_j u_j^{\mathrm{T}} \Phi(z) u_j \quad \textit{for} \quad z \geqslant 3\sqrt{R_{\max}}.$$

Then $\widehat{\alpha}_j$ has a unique real zero $\vartheta_j \in [3\sqrt{R_{\max}}, \infty)$. Moreover, this zero is simple and satisfies

$$\left| \vartheta_j - \lambda_j - \frac{\mathcal{V}_{jj}}{\lambda_j} \right| \leqslant 145 \frac{R_{\max}}{\lambda_j^3} \operatorname{osc}(\mathcal{R}).$$

Since $\widehat{\alpha}_j(\lambda_j)$ is increasing and $\widehat{\alpha}_j(\lambda_j) \leqslant 0$, we also have

$$\vartheta_j \geqslant \lambda_j. \tag{5.5}$$

The following result provides crude bounds on the operator norms of Green functions $G(z) = (zI - E)^{-1}$ and $G^{(k)} = (zI - E^{(k)})^{-1}$, where $E^{(k)}$ is obtained from E with the k-th row and column replaced by zero.

LEMMA 5.3 (Lemma 16 from [73])**.** *For $z \in \mathbb{C}$ with $|z| \geqslant 2\|E\|$, we have*

$$\|G(z)\| \leqslant \frac{2}{|z|}, \quad \|G^{(k)}\| \leqslant \frac{2}{|z|}. \tag{5.6}$$

LEMMA 5.4 (Bound on the noise matrix)**.** *Let $E = (E_{ij})_{n\times n}$ be a symmetric matrix whose entries are centered and independent (up to symmetry) random variables. Assume the entries E_{ij}'s are sub-Gaussian with $\|E_{ij}\|_{\psi_2} \leqslant K\sigma_{ij}$. Denote $R_{\max} = \max_i \sum_j \sigma_{ij}^2$. Then for any $D > 0$, we have*

$$\mathbb{P}\left(\|E\| \leqslant 2.9\sqrt{R_{\max}} + cK\sigma_{\max}(D+2)\log n \right) \geqslant 1 - 3n^{-D}$$

for an absolute constant $c > 0$.

The result is a direct consequence of the sharp non-asymptotic norm bounds established in Bandeira and van Handel [14]. We provide a short derivation in Appendix A.4.

6 Proof strategy and technical inputs

This section states the probabilistic inputs used in the proof of Theorem 3 and explains how they combine with the deterministic QVE estimates from Section 5. The two main probabilistic ingredients are the isotropic local laws for the resolvent $G(z)$ and the location theorem for the outlier eigenvalues of $A + E$.

6.1 Isotropic local laws

Consider $G(z) = (zI - E)^{-1}$ for $z \in \mathbb{C}^+$. Recall the QVE approximation $\Phi(z)$ defined in (5.1). Our first main technical step is to establish an isotropic local law for $G(z)$. Denote

$$\mathfrak{m}_D := \begin{cases} C'_{\text{gen}}(D+6)^{3/2}\beta K\sigma_{\max}(\log n)^2, & \text{under Assumption 1.2,} \\ C'_{\text{bd}}(D+6)(c_0+\beta_0)K\sigma_{\max}\log n, & \text{under Assumption 1.3.} \end{cases}$$

Here, C'_{gen} and C'_{bd} are sufficiently large absolute constants.

THEOREM 4 (Isotropic local law)**.** *Fix $D > 0$. Suppose Assumption 1.1 and one of Assumptions 1.2 and 1.3 hold. For any deterministic unit vectors $v, w \in \mathbb{R}^n$ and every fixed $z \in \mathbb{C}$ with $|z| \geqslant 6\sqrt{R_{\max}}$, we have*

$$\left|v^{\mathrm{T}}(G(z) - \Phi(z))w\right| \leqslant \frac{\mathfrak{m}_D}{|z|^2}$$

with probability at least $1 - n^{-D-4}$.

Theorem 4 is proved in Section 8. We also need a version that is uniform over the spectral domain where the outlier eigenvalues are located. Define

$$\mathcal{S}_{\text{out}} := \left\{ z \in \mathbb{C} : 6\sqrt{R_{\max}} \leqslant |z| \leqslant 2R_{\max}^3 \right\}.$$

This result is a direct consequence of Theorem 4; its proof is provided in Appendix E. Recall that $A = U\Lambda U^{\mathrm{T}}$.

COROLLARY 2 (Uniform compressed local law)**.** *Fix $D > 0$. Suppose the assumptions of Theorem 4 hold. Then, with probability at least $1 - n^{-D}$,*

$$\sup_{z \in \mathcal{S}_{\text{out}}} |z|^2 \|U^{\mathrm{T}}(G(z) - \Phi(z))U\| \leqslant C'r \cdot \mathfrak{m}_D = M_D, \tag{6.1}$$

$$\sup_{z \in \mathcal{S}_{\text{out}}} |z|^2 \max_{1 \leqslant i \leqslant n} \|e_i^{\mathrm{T}}(G(z) - \Phi(z))U\| \leqslant M_D. \tag{6.2}$$

Consequently,

$$\sup_{z \in \mathcal{S}_{\text{out}}} |z|^2 \max_{1 \leqslant i \leqslant n} \left\|e_i^{\mathrm{T}}Q(G(z) - \Phi(z))U\right\| \leqslant 2M_D, \tag{6.3}$$

where $Q = I - UU^{\mathrm{T}}$ and M_D is defined in (2.1).

6.2 Outlier eigenvalue location

We next state the outlier-location results used in the proof of the eigenspace perturbation results. For any $\lambda_j \geqslant 6\sqrt{R_{\max}}$, Proposition 5.1 guarantees that there exists a unique solution ϑ_j to the equation $\lambda_j^{-1} - u_j^{\mathrm{T}}\Phi(z)u_j = 0$ for $z \geqslant 3\sqrt{R_{\max}}$. Besides,

$$\vartheta_j \approx \lambda_j + \frac{\mathcal{V}_{jj}}{\lambda_j}.$$

For a simple supercritical eigenvalue λ_k, we will show that ϑ_k is the typical location of $\widetilde{\lambda}_k$.

Recall that

$$\rho_k \equiv \rho_{k,D} := 10M_D + 15\frac{\operatorname{osc}(\mathcal{R})}{\lambda_k}.$$

Denote the index set for the supercritical eigenvalues as

$$\mathcal{O}_+ := \{l \in [r_+] : \lambda_l \geqslant 6\sqrt{R_{\max}}\}.$$

THEOREM 5 (Individual outlier eigenvalue location)**.** *Fix $D > 0$ and $k \in [r_+]$. If $\lambda_k \geqslant 6\sqrt{R_{\max}} + 100\rho_k$ is a simple eigenvalue and the gap condition holds:*

$$\min_{l<k}(\vartheta_l - \vartheta_k) \geqslant 50\rho_k \quad \textit{and} \quad \min_{l\in\mathcal{O}_+,\, l>k}(\vartheta_k - \vartheta_l) \geqslant 50\rho_k$$

with the convention that the minimum over an empty set is $+\infty$, then, with probability at least $1 - n^{-D}$,

$$|\widetilde{\lambda}_k - \vartheta_k| \leqslant \rho_k.$$

The proof of Theorem 5 is given in Section 9. Furthermore, we provide the following result on the outlier eigenvalue clusters, which will be used in the proof of the top-k eigenspace perturbations.

THEOREM 6 (Lower edge of the top-k outlier clusters)**.** *Fix $D > 0$ and $k \in [r_+]$. Define*

$$\underline{\vartheta}_k := \min_{1\leqslant t\leqslant k} \vartheta_t.$$

If $\lambda_k \geqslant 6\sqrt{R_{\max}} + 100\rho_k$ and the gap condition holds:

$$\min_{l\in\mathcal{O}_+,\, l>k}(\underline{\vartheta}_k - \vartheta_l) \geqslant 50\rho_k \tag{6.4}$$

with the convention that the minimum over an empty set is $+\infty$, then, with probability at least $1 - n^{-D}$,

$$\widetilde{\lambda}_k \geqslant \underline{\vartheta}_k - \rho_k.$$

In particular, for every $s \in [k]$,

$$\widetilde{\lambda}_s \geqslant \underline{\vartheta}_k - \rho_k.$$

REMARK 6.1. *Theorem 6 holds under a stronger but more conventional assumption that the k-th deterministic root lower-bounds the preceding roots (i.e., $\vartheta_l \geqslant \vartheta_k$ for all $l < k$). Under that assumption, $\underline{\vartheta}_k = \vartheta_k$. However, we introduce $\underline{\vartheta}_k$ to avoid the internal ordering of these ϑ_s's for $s \leqslant k$. Indeed, the deterministic roots can physically cross and permute, even when the underlying λ_s's are strictly ordered. For instance, even if $\lambda_l > \lambda_s$ for $l < s \in [k]$, as given in Proposition 5.1, $\vartheta_l \approx \lambda_l + \frac{\mathcal{V}_{ll}}{\lambda_l}$ could be dominated by $\vartheta_s \approx \lambda_s + \frac{\mathcal{V}_{ss}}{\lambda_s}$. This is because the shift $\mathcal{V}_{ll} = u_l^{\mathrm{T}} \mathcal{R} u_l$ depends on the alignment of the l-th population eigenvector with the variance profile $\mathcal{R}$. In the presence of anisotropic noise, it is entirely possible for a lower-ranked spike's variance projection (e.g., $\mathcal{V}_{ss}$) to be significantly larger than that of a higher-ranked spike (e.g., $\mathcal{V}_{ll}$).*

The proof of Theorem 6 parallels that of Theorem 5 and can be found in Section 9.

6.3 Proof roadmap

We briefly explain how the technical inputs above are used to prove Theorem 3. The proof is carried out on a high-probability event where $\|E\| \leqslant 3\sqrt{R_{\max}}$ and the isotropic local laws Theorem 4 and Corollary 2 hold.

The signal condition $\lambda_k \geqslant 9\sqrt{R_{\max}}$ ensures that the outlier eigenvalues satisfy

$$\widetilde{\lambda}_s \geqslant \lambda_k - \|E\| \geqslant 6\sqrt{R_{\max}} \tag{6.5}$$

for all $s \in [k]$ by Weyl's inequalty. Thus the uniform local law Corollary 2 applies at the random eigenvalues $z = \widetilde{\lambda}_s$.

The deterministic part of the proof starts from the eigenvector equation

$$\widetilde{u}_s = G(\widetilde{\lambda}_s) A \widetilde{u}_s, \qquad s \in [k].$$

Projecting this identity onto the non-target signal directions and the null space gives separate bounds for how much $\widetilde{u}_s$ extends outside the target space. The null-space projection produces the usual signal-to-noise contribution $\|E\|/\lambda_k$. For the signal-space terms, we decompose

$$G(z) = \Phi(z) + \Xi(z), \qquad \Xi(z) := G(z) - \Phi(z).$$

The compressed local law controls the contribution of $\Xi(z)$, while the large-z expansion of $\Phi(z)$ identifies the deterministic variance-profile contribution. This yields the stochastic term involving M_D and the geometric bias term $\mathcal{B}_k$. The final operator-norm and $2 \to \infty$ estimates follow from these blockwise bounds through deterministic reductions.

7 Proof of the main perturbation bounds: Theorem 3

Set

$$\Xi(z) := G(z) - \Phi(z).$$

Fix $D > 0$. Define the events

$$\mathcal{E}_{\mathrm{op}} := \{\|E\| \leqslant 3\sqrt{R_{\max}}\}$$

and

$$\mathcal{E}_{\mathrm{loc}}(D) := \left\{ \sup_{z\in\mathcal{S}} |z|^2 \|U^{\mathrm{T}}\Xi(z)U\| \leqslant M_D \right\} \cap \left\{ \sup_{z\in\mathcal{S}_{\mathrm{out}}} |z|^2 \max_{1\leqslant i\leqslant n} \left\|e_i^{\mathrm{T}} Q\Xi(z)U\right\| \leqslant 2M_D \right\},$$

where $\mathcal{S}_{\mathrm{out}} := \left\{z \in \mathbb{C} : 6\sqrt{R_{\max}} \leqslant |z| \leqslant 2R_{\max}^3\right\}$ and $Q = I - UU^{\mathrm{T}}$. Set

$$\mathcal{E}_D := \mathcal{E}_{\mathrm{op}} \cap \mathcal{E}_{\mathrm{loc}}(D).$$

By Lemma 5.4 and Corollary 2, we have

$$\mathbb{P}(\mathcal{E}_D) \geqslant 1 - n^{-D}.$$

All estimates below are deterministic on $\mathcal{E}_D$, except when the outlier-location theorem is invoked. In that step we intersect with the corresponding outlier-location event, which still has probability at least $1 - n^{-D}$ after adjusting constants.

On the event $\mathcal{E}_D$, note that our signal assumption $9\sqrt{R_{\max}} \leqslant \lambda_k \leqslant R_{\max}^3$ ensures $\widetilde{\lambda}_s \in \mathcal{S}_{\mathrm{out}}$ for all $s \leqslant k$ by Weyl's inequality (see (6.5)).

For simplicity, we write

$$M \equiv M_D.$$

We first present deterministic bounds on the difference between the eigenspaces U_k and $\widetilde{U}_k$. Let $J = [r]\backslash[k]$ denote the complement index set and $U_J = (u_{k+1}, \ldots, u_r)$. Let P_W be the orthogonal projection onto the subspace W.

Proposition 7.1. *For $1 \leqslant k \leqslant r+$, assume $\lambda_k \geqslant 2\|E\|$. Then the following deterministic bounds hold.*

(i) *For the operator norm bound, we have*

$$\|\sin\angle(U_k, \widetilde{U}_k)\| \leqslant \|U_J^{\mathrm{T}}\widetilde{U}_k\|_F + 2\frac{\|E\|}{\lambda_k}. \tag{7.1}$$

(ii) *For the $2 \to \infty$ norm bound, we have*

$$\min_{O\in\mathbb{O}(k)} \|\widetilde{U}_k - U_k O\|_{2,\infty} \leqslant \|\widetilde{U}_k - P_{U_k}\widetilde{U}_k\|_{2,\infty} + \|U_k\|_{2,\infty}\|\sin\angle(U_k, \widetilde{U}_k)\|^2, \tag{7.2}$$

where

$$\begin{aligned}\|\widetilde{U}_k - P_{U_k}\widetilde{U}_k\|_{2,\infty} \leqslant{}& \|U\|_{2,\infty} \cdot \|U_J^{\mathrm{T}}\widetilde{U}_k\|_F + \frac{13}{2}\sqrt{k}\|U\|_{2,\infty}\frac{R_{\max}}{\widetilde{\lambda}_k^2} \\ &+ 4\sqrt{\sum_{s=1}^{k} \widetilde{\lambda}_s^2 \max_{1\leqslant i\leqslant n} \left\|e_i^{\mathrm{T}} Q\Xi(z)U\right\|^2}.\end{aligned} \tag{7.3}$$

Given these deterministic bounds, the main technical contribution of our perturbation analysis is the estimation of

$$\|U_J^{\mathrm{T}}\widetilde{U}_k\|_F = \sqrt{\sum_{s=1}^{k} \|U_J^{\mathrm{T}}\widetilde{u}_s\|^2}.$$

We first present a deterministic bound for each $\|U_J^{\mathrm{T}}\widetilde{u}_s\|$. The geometric bias matrix $\mathcal{V} = U^{\mathrm{T}}\mathcal{R}U$ is defined in (1.2). For any index sets $\mathtt{I}, \mathtt{J} \in [r]$ with $U_{\mathtt{I}} = (u_i)_{i\in\mathtt{I}}$ and $U_{\mathtt{J}} = (u_j)_{j\in\mathtt{J}}$, we denote

$$\mathcal{V}_{\mathtt{IJ}} := U_{\mathtt{I}}^{\mathrm{T}}\mathcal{R}\,U_{\mathtt{J}}.$$

For a matrix B, we denote by $\mathrm{Off}(B)$ the matrix obtained by setting all diagonal entries to zero.

PROPOSITION 7.2. *Define index sets*

$$\mathcal{J} := \{k+1, \ldots, r_+\}, \quad \mathcal{I} := [r]\backslash\mathcal{J},$$
$$\mathcal{N} := \{r_+ + 1, \ldots, r\}, \quad \mathcal{K} := [r]\backslash\mathcal{N}.$$

Fix $s \in [k]$. Define the gap

$$\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s) := \widetilde{\lambda}_s \min_{j\in\mathcal{J}} |1 - \lambda_j u_j^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)u_j|$$

with the convention that the minimum over an empty set is $+\infty$. If the following gap condition holds:

$$\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s) \geqslant \frac{2\lambda_{k+1}}{\widetilde{\lambda}_s^2}\|\mathrm{Off}(\mathcal{V}_{\mathcal{J}\mathcal{J}})\| + \frac{13\lambda_{k+1}R_{\max}}{\widetilde{\lambda}_s^4}\mathrm{osc}(\mathcal{R}), \tag{7.4}$$

then we have

$$\begin{aligned}\|U_J^{\mathrm{T}}\widetilde{u}_s\| \leqslant &\frac{4}{\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s)}\left(\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\widetilde{\lambda}_s} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^3}\mathrm{osc}(\mathcal{R}) + \widetilde{\lambda}_s^2\|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right) \\ &+ 2\sqrt{3}\left(\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\widetilde{\lambda}_s^2} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^4}\mathrm{osc}(\mathcal{R}) + \widetilde{\lambda}_s\|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right).\end{aligned} \tag{7.5}$$

Note that when $\mathcal{J} = \varnothing$, we have $\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s) = \infty$, so the corresponding term on the right-hand side of (7.5) vanishes. The proofs of Propositions 7.1 and 7.2 are deferred to Appendix B. On the event $\mathcal{E}_D$, the condition $\lambda_k \geqslant 9\sqrt{R_{\max}}$ implies $\lambda_k \geqslant 3\|E\| \geqslant 2\|E\|$, so Proposition 7.1 applies.

Now we continue with the estimation of $\|U_J^{\mathrm{T}}\widetilde{u}_s\|$ and split the discussion into two cases: $k = r_+$ or $k < r_+$.

Case 1. Assume $k = r_+$. Then $\mathcal{J} = \varnothing$ and $\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s) = +\infty$. In this case,

$$J = \mathcal{N} = \{r_+ + 1, \ldots, r\}.$$

We just bound

$$\|U_J^{\mathrm{T}}\widetilde{u}_s\| \leqslant 2\sqrt{3}\left(\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\widetilde{\lambda}_s^2} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^4}\mathrm{osc}(\mathcal{R}) + \widetilde{\lambda}_s\|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right).$$

By Theorem 6 (note that the gap condition (6.4) holds trivially), for every $s \in [k]$,

$$\widetilde{\lambda}_s \geqslant \underline{\vartheta}_k - \rho_k \geqslant \lambda_k - \rho_k \geqslant \frac{99}{100}\lambda_k.$$

Thus,

$$\|U_J^{\mathrm{T}}\widetilde{u}_s\| = \|U_{\mathcal{N}}^{\mathrm{T}}\widetilde{u}_s\| < 4\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 24\frac{R_{\max}}{\lambda_k^4}\mathrm{osc}(\mathcal{R}) + 4\frac{M}{\lambda_k} \tag{7.6}$$

and

$$\|U_J^{\mathrm{T}}\widetilde{U}_k\|_F = \sqrt{\sum_{s=1}^{k}\|U_J^{\mathrm{T}}\widetilde{u}_s\|^2} \leqslant \sqrt{r_+}\left(4\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_{r_+}^2} + 24\frac{R_{\max}}{\lambda_{r_+}^4}\mathrm{osc}(\mathcal{R}) + 4\frac{M}{\lambda_{r_+}}\right). \tag{7.7}$$

<u>Case 2.</u> Assume $k < r_+$. Under our gap assumption

$$\delta_k \geqslant 65\rho_k + 20\frac{\mathrm{osc}(\mathcal{R})}{\lambda_k} \geqslant 65\rho_k + 20\frac{\lambda_{k+1}}{\lambda_k^2}\mathrm{osc}(\mathcal{R}), \tag{7.8}$$

we first show that (6.4) holds, that is,

$$\min_{l\in\mathcal{O}_+,\, l>k}(\underline{\vartheta}_k - \vartheta_l) \geqslant 50\rho_k \quad \text{where} \quad \underline{\vartheta}_k = \min_{1\leqslant t\leqslant k}\vartheta_t.$$

If $\mathcal{O}_+\backslash[k] = \varnothing$, then the inequality holds because the left-hand side is $+\infty$ by our notation. Assume $\mathcal{O}_+\backslash[k] \neq \varnothing$.

Suppose

$$\underline{\vartheta}_k = \vartheta_t$$

for some $1 \leqslant t \leqslant k$. Observe from (5.5) that

$$\vartheta_t \geqslant \lambda_t \geqslant \lambda_k > \lambda_{k+1} \geqslant \lambda_j \tag{7.9}$$

for any $j \in \mathcal{J} = \{k+1, \ldots, r_+\}$.

Define two auxiliary functions

$$\mathcal{L}(x) := x + \frac{R_{\min}}{x} - 145\frac{R_{\max}}{x^3}\mathrm{osc}(\mathcal{R}), \quad \mathcal{U}(x) := x + \frac{R_{\max}}{x} + 145\frac{R_{\max}}{x^3}\mathrm{osc}(\mathcal{R})$$

for $x \geqslant 6\sqrt{R_{\max}}$. Note that both $\mathcal{L}(x)$ and $\mathcal{U}(x)$ are increasing functions. Note that $\lambda_t \geqslant 9\sqrt{R_{\max}}$. Thus,

$$\mathcal{L}(\lambda_t) \leqslant \vartheta_t \leqslant \mathcal{U}(\lambda_t).$$

For any $l \in \mathcal{O}_+$ satisfying $l > k$, since $\lambda_l \leqslant \lambda_{k+1}$, by the definition of ϑ_l for and Proposition 5.1, we have

$$\mathcal{L}(\lambda_l) \leqslant \vartheta_l \leqslant \mathcal{U}(\lambda_l) \leqslant \mathcal{U}(\lambda_{k+1}).$$

In particular, for any $l \in \mathcal{O}_+\backslash[k]$ (note that $t \in \mathcal{O}_+$), we have

$$\begin{aligned}\vartheta_t - \vartheta_l &\geqslant \mathcal{L}(\lambda_t) - \mathcal{U}(\lambda_{k+1}) \\ &= (\lambda_t - \lambda_{k+1}) + \frac{R_{\min}}{\lambda_t} - \frac{R_{\max}}{\lambda_{k+1}} - 145\frac{R_{\max}}{\lambda_t^3}\mathrm{osc}(\mathcal{R}) - 145\frac{R_{\max}}{\lambda_{k+1}^3}\mathrm{osc}(\mathcal{R}).\end{aligned}$$

Rewriting

$$\frac{R_{\min}}{\lambda_t} - \frac{R_{\max}}{\lambda_{k+1}} = \frac{\lambda_{k+1} - \lambda_t}{\lambda_t \lambda_{k+1}} R_{\min} + \frac{R_{\min} - R_{\max}}{\lambda_{k+1}}$$

and using the assumption $\lambda_t, \lambda_{k+1} \geqslant 9\sqrt{R_{\max}} \geqslant 6\sqrt{R_{\min}}$, we further get

$$\begin{aligned}
\vartheta_t - \vartheta_l &\geqslant \left(1 - \frac{R_{\min}}{\lambda_t \lambda_{k+1}}\right)(\lambda_t - \lambda_{k+1}) + \frac{R_{\min} - R_{\max}}{\lambda_{k+1}} - \frac{145}{36}\frac{\operatorname{osc}(\mathcal{R})}{\lambda_t} - \frac{145}{36}\frac{\operatorname{osc}(\mathcal{R})}{\lambda_{k+1}} \\
&\geqslant \frac{35}{36}\delta_k - \left(1 + \frac{145}{36}\right)\frac{\operatorname{osc}(\mathcal{R})}{\lambda_{k+1}} - \frac{145}{36}\frac{\operatorname{osc}(\mathcal{R})}{\lambda_t} \\
&\geqslant \frac{35}{36}\delta_k - \frac{181}{36}\frac{\operatorname{osc}(\mathcal{R})}{\lambda_{k+1}} - \frac{145}{36}\frac{\operatorname{osc}(\mathcal{R})}{\lambda_k},
\end{aligned}$$

where we used $\lambda_t \geqslant \lambda_k$ in the last inequality. To proceed, note that

$$\frac{\operatorname{osc}(\mathcal{R})}{\lambda_{k+1}} = \frac{\operatorname{osc}(\mathcal{R})}{\lambda_k} + \delta_k \frac{\operatorname{osc}(\mathcal{R})}{\lambda_k \lambda_{k+1}} \leqslant \frac{\operatorname{osc}(\mathcal{R})}{\lambda_k} + \frac{1}{36}\delta_k.$$

We get

$$\vartheta_t - \vartheta_l \geqslant \left(\frac{35}{36} - \frac{181}{36^2}\right)\delta_k - \frac{326}{36}\frac{\operatorname{osc}(\mathcal{R})}{\lambda_k} \geqslant 50\rho_k$$

by our assumption (7.8) on δ_k. Hence, the conclusion of Theorem 6 holds. In particular, for every $s \in [k]$,

$$\widetilde{\lambda}_s \geqslant \underline{\vartheta}_k - \rho_k = \vartheta_t - \rho_k. \tag{7.10}$$

Recall $\Delta^{\Phi}_{\mathcal{J}}$ defined in Proposition 7.2. Next, we show that for any $s \in [k]$,

$$\Delta^{\Phi}_{\mathcal{J}}(\widetilde{\lambda}_s) = \widetilde{\lambda}_s \min_{j \in \mathcal{J}} |1 - \lambda_j u_j^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) u_j| \geqslant \frac{1}{2}\delta_k.$$

Denote $\alpha_j(z) = \lambda_j^{-1} - u_j^{\mathrm{T}}\Phi(z)u_j$ for each $j \in [r]$. Then

$$\Delta^{\Phi}_{\mathcal{J}}(\widetilde{\lambda}_s) = \min_{j \in \mathcal{J}} \left|\widetilde{\lambda}_s \lambda_j \alpha_j(\widetilde{\lambda}_s)\right|.$$

For each $j \in \mathcal{J}$, by the expansion of $\Phi(z)$ in (5.3), we have

$$\alpha_j(z) = \frac{1}{\lambda_j} - u_j^{\mathrm{T}}\Phi(z)u_j = \frac{1}{\lambda_j} - \frac{1}{z} - \frac{\mathcal{V}_{jj}}{z^3} - \varepsilon_{jj}(z),$$

where $\varepsilon_{jj}(z) := u_j^{\mathrm{T}}\varepsilon(z)u_j$. Similarly,

$$\alpha_t(z) = \frac{1}{\lambda_t} - \frac{1}{z} - \frac{\mathcal{V}_{tt}}{z^3} - \varepsilon_{tt}(z).$$

From the difference $\alpha_j(z) - \alpha_t(z)$, we find

$$\alpha_j(z) = \left(\frac{1}{\lambda_j} - \frac{1}{\lambda_t}\right) + \alpha_t(z) - \frac{\mathcal{V}_{jj} - \mathcal{V}_{tt}}{z^3} - (\varepsilon_{jj}(z) - \varepsilon_{tt}(z)).$$

Therefore,

$$\widetilde{\lambda}_s\lambda_j\alpha_j(\widetilde{\lambda}_s) = \widetilde{\lambda}_s\left(1-\frac{\lambda_j}{\lambda_t}\right) + \widetilde{\lambda}_s\lambda_j\alpha_t(\widetilde{\lambda}_s) - \frac{\lambda_j}{\widetilde{\lambda}_s^2}\left(\mathcal{V}_{jj}-\mathcal{V}_{tt}\right) - \widetilde{\lambda}_s\lambda_j(\varepsilon_{jj}(\widetilde{\lambda}_s)-\varepsilon_{tt}(\widetilde{\lambda}_s))$$
$$:= T_1 + T_2 + T_3 + T_4.$$

It follows that

$$\left|\widetilde{\lambda}_s\lambda_j\alpha_j(\widetilde{\lambda}_s)\right| \geqslant T_1 + T_2 - |T_3| - |T_4|.$$

We estimate T_i's respectively. For T_1, by (7.9), we have

$$T_1 = \widetilde{\lambda}_s\left(1-\frac{\lambda_j}{\lambda_t}\right) \geqslant \widetilde{\lambda}_s\left(1-\frac{\lambda_{k+1}}{\lambda_k}\right) = \frac{\widetilde{\lambda}_s}{\lambda_k}\delta_k.$$

Then, by (7.10), we get

$$T_1 \geqslant (\vartheta_t-\rho_k)\frac{\delta_k}{\lambda_k} = \frac{\vartheta_t}{\lambda_k}\delta_k - \rho_k\frac{\delta_k}{\lambda_k} \geqslant \delta_k - \rho_k$$

since $\frac{\vartheta_t}{\lambda_k} \geqslant 1$ by (7.9) and $\frac{\delta_k}{\lambda_k} \leqslant 1$.

Next, we estimate $T_2 = \widetilde{\lambda}_s\lambda_j\alpha_t(\widetilde{\lambda}_s)$. By the same estimates as (9.20) and (9.21), we have

$$\frac{5}{2x^2} \leqslant \alpha_t'(x) \leqslant \frac{2}{x^2} \quad \text{for } x \geqslant 3\sqrt{R_{\max}}.$$

In particular, $\alpha_t(x)$ is increasing for $x \geqslant 3\sqrt{R_{\max}}$. By our definition, $\alpha_t(\vartheta_t)$=0. Since $\widetilde{\lambda}_s \geqslant \vartheta_t - \rho_k$ by (7.10),

$$\alpha_t(\widetilde{\lambda}_s) \geqslant \alpha_t(\vartheta_t-\rho_k) = \alpha_t(\vartheta_t-\rho_k) - \alpha_t(\vartheta_t) = -\rho_k\alpha_t'(\zeta),$$

where the last equation is due to the mean value theorem and ζ is a value between $\vartheta_t - \rho_k$ and ϑ_t. Then, by (7.9),

$$\widetilde{\lambda}_s\lambda_j\alpha_t(\widetilde{\lambda}_s) \geqslant -\widetilde{\lambda}_s\lambda_j\frac{2}{\zeta^2}\rho_k \geqslant -2\frac{\widetilde{\lambda}_s\lambda_{k+1}}{(\vartheta_t-\rho_k)^2}\rho_k.$$

Since $\vartheta_t - \rho_k \geqslant \frac{99}{100}\vartheta_t$ and $\widetilde{\lambda}_s \leqslant \frac{3}{2}\lambda_s \leqslant \frac{3}{2}\vartheta_s$ by Weyl's inequality and (5.5), we have

$$T_2 \geqslant -2(\frac{100}{99})^2\frac{\frac{3}{2}\vartheta_s\lambda_{k+1}}{\vartheta_t^2}\rho_k > -4\rho_k.$$

Finally, for T_3 and T_4, by Lemma 5.2, the bounds $\widetilde{\lambda}_s \leqslant \frac{3}{2}\lambda_s$ and

$$\widetilde{\lambda}_s \geqslant \vartheta_t - \rho_k \geqslant \frac{99}{100}\vartheta_t \geqslant \frac{99}{100}\lambda_k,$$

we have

$$|T_3| + |T_4| \leqslant \frac{\lambda_j}{\widetilde{\lambda}_s^2}\mathrm{osc}(\mathcal{R}) + \widetilde{\lambda}_s\lambda_j\cdot\mathrm{osc}(\varepsilon(\widetilde{\lambda}_s))$$
$$\leqslant 4\frac{\lambda_{k+1}}{\lambda_k^2}\mathrm{osc}(\mathcal{R}) + \frac{15\cdot 2^4}{36}\frac{\lambda_{k+1}}{\lambda_k^2}\mathrm{osc}(\mathcal{R}) = \frac{32}{3}\frac{\lambda_{k+1}}{\lambda_k^2}\mathrm{osc}(\mathcal{R}).$$

Combining the above estimates, we arrive at

$$\widetilde{\lambda}_s \lambda_j \alpha_j(\widetilde{\lambda}_s) \geqslant \delta_k - 5\rho_k - \frac{32}{3}\frac{\lambda_{k+1}}{\lambda_k^2}\mathrm{osc}(\mathcal{R}) \geqslant \frac{1}{2}\delta_k$$

by our gap assumption (7.8). Hence,

$$\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s) = \min_{j\in\mathcal{J}} \left|\widetilde{\lambda}_s \lambda_j \alpha_j(\widetilde{\lambda}_s)\right| \geqslant \frac{1}{2}\delta_k.$$

Furthermore, (7.4) in Proposition 7.2 holds. To see this, note that the right-hand side of (7.2)

$$\begin{aligned}
&\frac{2\lambda_{k+1}}{\widetilde{\lambda}_s^2}\|\mathrm{Off}(\mathcal{V}_{\mathcal{J}\mathcal{J}})\| + \frac{13\lambda_{k+1}R_{\max}}{\widetilde{\lambda}_s^4}\mathrm{osc}(\mathcal{R}) \\
&\leqslant 2\left(\frac{100}{99}\right)^4 \frac{\lambda_{k+1}}{\lambda_k^2}\mathrm{osc}(\mathcal{R}) + \frac{13}{36}\left(\frac{100}{99}\right)^4 \frac{\lambda_{k+1}}{\lambda_k^2}\mathrm{osc}(\mathcal{R}) \\
&< 3\frac{\lambda_{k+1}}{\lambda_k^2}\mathrm{osc}(\mathcal{R}) < \frac{1}{2}\delta_k \leqslant \Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s).
\end{aligned}$$

We continue from the bound established in Proposition 7.2. On the event $\mathcal{E}_D$, substituting the uniform local law bound $\|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\| \leqslant M/\widetilde{\lambda}_s^2$ and using $\widetilde{\lambda}_s \geqslant \frac{99}{100}\lambda_k$,

$$\begin{aligned}
\|U_J^{\mathrm{T}}\widetilde{u}_s\| \leqslant &\frac{4}{\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s)}\left(\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\widetilde{\lambda}_s} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^2}\mathrm{osc}(\mathcal{R}) + \widetilde{\lambda}_s^2\|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right) \\
&+ 2\sqrt{3}\left(\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\widetilde{\lambda}_s^2} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^4}\mathrm{osc}(\mathcal{R}) + \widetilde{\lambda}_s\|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right) \\
&\leqslant \frac{8}{\delta_k}\left(\frac{100}{99}\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\lambda_k} + \frac{13}{2}\left(\frac{100}{99}\right)\frac{\mathrm{osc}(\mathcal{R})R_{\max}}{\lambda_k^2} + M\right) \\
&+ 2\sqrt{3}\left(\left(\frac{100}{99}\right)^2\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + \frac{13}{2}\left(\frac{100}{99}\right)\frac{\mathrm{osc}(\mathcal{R})R_{\max}}{\lambda_k^2} + \frac{100}{99}\frac{M}{\lambda_k}\right)
\end{aligned}$$

Now the bound above simplifies to

$$\|U_J^{\mathrm{T}}\widetilde{u}_s\| < \frac{10}{\delta_k\lambda_k}\left(\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| + 8\frac{R_{\max}}{\lambda_k^2}\mathrm{osc}(\mathcal{R})\right) + 3\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 10\frac{M}{\delta_k}.$$

Finally, we establish the bound

$$\begin{aligned}
\|U_J^{\mathrm{T}}\widetilde{U}_k\|_F &= \sqrt{\sum_{s=1}^{k}\|U_J^{\mathrm{T}}\widetilde{u}_s\|^2} \\
&\leqslant \frac{10\sqrt{k}}{\delta_k\lambda_k}\left(\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| + 8\frac{R_{\max}}{\lambda_k^2}\mathrm{osc}(\mathcal{R})\right) + 3\sqrt{k}\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 10\sqrt{k}\frac{M}{\delta_k}.
\end{aligned}$$

Combining (7.7), for any $k \in [r_+]$, we always have

$$\|U_J^{\mathrm{T}}\widetilde{U}_k\|_F \leqslant \mathcal{B}_k + 10\sqrt{k}\frac{M}{\delta_k}, \tag{7.11}$$

where

$$\mathcal{B}_k := \frac{10\sqrt{k}}{\delta_k \lambda_k}\left(\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| + 8\frac{R_{\max}}{\lambda_k^2}\operatorname{osc}(\mathcal{R})\right) + 4\sqrt{k}\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2}.$$

Consequently, by (7.1),

$$\|\sin\angle(U_k, \widetilde{U}_k)\| \leqslant \mathcal{B}_k + 10\sqrt{k}\frac{M}{\delta_k} + 2\frac{\|E\|}{\lambda_k}.$$

From (7.3), using (7.11), on the event $\mathcal{E}_D$, we also have

$$\begin{aligned}
&\|\widetilde{U}_k - P_{U_k}\widetilde{U}_k\|_{2,\infty} \\
&\leqslant \|U\|_{2,\infty}\cdot\|U_J^{\mathrm{T}}\widetilde{U}_k\|_F + \frac{13}{2}\sqrt{k}\|U\|_{2,\infty}\frac{R_{\max}}{\widetilde{\lambda}_k^2} + 4\sqrt{\sum_{s=1}^{k}\widetilde{\lambda}_s^2 \max_{1\leqslant i\leqslant n}\left\|e_i^{\mathrm{T}}Q\Xi(z)U\right\|^2} \\
&\leqslant \|U\|_{2,\infty}\left(\mathcal{B}_k + 10\sqrt{k}\frac{M}{\delta_k} + 8\frac{R_{\max}}{\lambda_k^2}\right) + 4M\sqrt{\sum_{s=1}^{k}\frac{1}{\widetilde{\lambda}_s^2}} \\
&\leqslant \|U\|_{2,\infty}\left(\mathcal{B}_k + 10\sqrt{k}\frac{M}{\delta_k} + 8\frac{R_{\max}}{\lambda_k^2}\right) + 5\sqrt{k}\frac{M}{\lambda_k}.
\end{aligned}$$

Now, from (7.2), we obtain that

$$\begin{aligned}
&\min_{O\in\mathbb{O}(k)}\|\widetilde{U}_k - U_kO\|_{2,\infty} \\
&\leqslant \|\widetilde{U}_k - P_{U_k}\widetilde{U}_k\|_{2,\infty} + \|U_k\|_{2,\infty}\|\sin\angle(U_k,\widetilde{U}_k)\|^2 \\
&\leqslant 4\|U\|_{2,\infty}\left(\mathcal{B}_k + 20\sqrt{k}\frac{M}{\delta_k} + 30\frac{R_{\max}}{\lambda_k^2}\right) + 120\sqrt{k}\frac{M}{\lambda_k}
\end{aligned}$$

by our assumptions on λ_k and δ_k.

8 Proof of the isotropic local law: Theorem 4

Let $z \in \mathbb{C}$ satisfy $|z| \geqslant 6\sqrt{R_{\max}}$. Recall that

$$G(z) = (zI - E)^{-1}.$$

Denote

$$H \equiv H(z) = \operatorname{diag}(h_1(z), \ldots, h_n(z)), \qquad h_i \equiv h_i(z) = \sum_{j=1}^{n}\sigma_{ij}^2\phi_j(z).$$

Then the QVE can be written as

$$\Phi(z) = (zI - H)^{-1}.$$

Since $(zI - E)G = I$ and $(zI - H)\Phi = I$, subtracting these identities, we have

$$z(G - \Phi) = EG - H\Phi = (E - H)G + H(G - \Phi).$$

Rearranging the terms, we further obtain $(zI-H)(G-\Phi) = (E-H)G$. Hence, for deterministic unit vectors $v, w \in \mathbb{R}^n$,

$$v^{\mathrm{T}}(G-\Phi)w = v^{\mathrm{T}}(z-H)^{-1}(E-H)Gw. \tag{8.1}$$

Set

$$\widehat{v}^{\mathrm{T}} := v^{\mathrm{T}}(z-H)^{-1}.$$

Then

$$v^{\mathrm{T}}(G-\Phi)w = \widehat{v}^{\mathrm{T}}(E-H)Gw.$$

We also define

$$L \equiv L(z) = \operatorname{diag}(l_1(z), \ldots, l_n(z)), \qquad l_i \equiv l_i(z) = \sum_{j=1}^{n} \sigma_{ij}^2 G_{jj}(z).$$

Then (8.1) gives the decomposition

$$v^{\mathrm{T}}(G-\Phi)w = \widehat{v}^{\mathrm{T}}(E-L)Gw + \widehat{v}^{\mathrm{T}}(L-H)Gw. \tag{8.2}$$

We start with a bound on $\|\widehat{v}\|$. By Lemma 5.1,

$$\|H\| \leqslant R_{\max} \max_i |\phi_i(z)| \leqslant \frac{3R_{\max}}{2|z|} \leqslant \frac{|z|}{4},$$

where the last inequality follows from $|z|^2 \geqslant 36R_{\max}$. Hence

$$\|\widehat{v}\| \leqslant \|(zI-H)^{-1}\| \leqslant \frac{1}{|z|-\|H\|} \leqslant \frac{4}{3|z|}. \tag{8.3}$$

With this $\|\widehat{v}\|$ bound, the control of the first error term in (8.2) reduces to the following proposition, which we prove in Appendix C. The proof follows the cumulant expansion strategy of He, Knowles and Rosenthal [57], which compares the random resolvent with its deterministic self-consistent approximation. Our application is simpler in two ways. First, we work only in the large-z outlier regime, where resolvent derivatives have elementary bounds. Second, we need only a second-order cumulant expansion with a controlled third-order remainder. The main challenge is handling highly inhomogeneous and sparse variance profiles.

PROPOSITION 8.1. *Fix $D > 0$. Let $v, w \in \mathbb{R}^n$ be deterministic unit vectors, and let $z \in \mathbb{C}_+$ satisfy $|z| \geqslant 6\sqrt{R_{\max}}$. Under Assumption 1.1 and one of Assumptions 1.2 and 1.3, with probability at least $1-n^{-D-4}$,*

$$\left|v^{\mathrm{T}}(E-L)Gw\right| \leqslant \frac{\mathfrak{m}_D}{|z|} + Cn^{-500}.$$

Here, the Cn^{-500} term appears only in the general sub-Gaussian regime; in the bounded sparse-entry regime, it may be omitted.

The second error term in (8.2) is controlled using Proposition 8.1 and a deterministic stability argument for the diagonal self-consistent equation. Its proof is given in Appendix D.

PROPOSITION 8.2. *Fix $D > 0$. Under the assumption of Propositon 8.1, with probability at least $1 - n^{-D-4}$:*

$$\left|v^{\mathrm{T}}(L-H)Gw\right| \leqslant C\left(\frac{\mathfrak{m}_D}{|z|} + n^{-500}\right).$$

In the bounded sparse-entry regime, the n^{-500} term may be omitted.

Proof of Theorem 4. We use the decomposition (8.2). For the first error term in (8.2), Proposition 8.1 and (8.3) yield

$$\left|\widehat{v}^{\mathrm{T}}(E-L)Gw\right| \leqslant \|\widehat{v}\|\left(\frac{\mathfrak{m}_D}{|z|} + Cn^{-500}\right) \leqslant C\frac{\mathfrak{m}_D}{|z|^2} + C\frac{n^{-500}}{|z|}. \tag{8.4}$$

Similarly, for the second error term in (8.2), from Proposition 8.2 and (8.3), we get

$$\left|\widehat{v}^{\mathrm{T}}(L-H)Gw\right| \leqslant C\frac{\mathfrak{m}_D}{|z|^2} + C\frac{n^{-500}}{|z|}. \tag{8.5}$$

Combining (8.4) and (8.5) with (8.2), we obtain

$$\left|v^{\mathrm{T}}(G-\Phi)w\right| \leqslant C\frac{\mathfrak{m}_D}{|z|^2} + C\frac{n^{-500}}{|z|}.$$

Since $R_{\max} \leqslant n^{100}$ and $\sigma_{\max}^{-1} \leqslant n^{100}$, we have $\mathfrak{m}_D \geqslant cn^{-100}$ for an absolute constant $c > 0$. Absorbing the n^{-500}-term, we get

$$\left|v^{\mathrm{T}}(G-\Phi)w\right| \leqslant \frac{\mathfrak{m}_D}{|z|^2}.$$

This completes the proof. □

9 Proofs of the outlier eigenvalue locations: Theorems 5 and 6

9.1 Proof of Theorem 5

For notational convenience throughout this section, we replace the target index k with j. We first consider the case that $\lambda_j \geqslant 6\sqrt{R_{\max}}$ for $j \in [r_+]$ is a simple eigenvalue. In this section, we use the normalized version

$$\alpha_j(z) := \lambda_j^{-1} - u_j^{\mathrm{T}}\Phi(z)u_j,$$

so that $\widehat{\alpha}_j(z) = \lambda_j\alpha_j(z)$. The deterministic root ϑ_j defined in Proposition 5.1 is equivalently characterized by $\alpha_j(\vartheta_j) = 0$.

We still work on the event $\mathcal{E}_D$. However, we only need a weaker signal assumption $\lambda_j \geqslant 6\sqrt{R_{\max}} + 100\rho_j$ since the local law is applied on deterministic contours around λ_j, whereas in the eigenspace proof, it is applied at the random points $z = \widetilde{\lambda}_s$.

By Proposition 5.1, there exists a unique deterministic location ϑ_j satisfying

$$1 - \lambda_j u_j^{\mathrm{T}}\Phi(\vartheta_j)u_j = 0$$

with

$$\left|\vartheta_j - \lambda_j - \frac{\mathcal{V}_{jj}}{\lambda_j}\right| \leqslant 145\frac{R_{\max}^2}{\lambda_j^3}.$$

In particular, since $\mathcal{V}_{jj} \geqslant 0$, we have

$$\vartheta_j \leqslant \lambda_j + \frac{\mathcal{V}_{jj}}{\lambda_j} + 145\frac{R_{\max}^2}{\lambda_j^3} \leqslant \lambda_j + \frac{1}{36}\lambda_j + \frac{145}{36^2}\lambda_j < 1.16\lambda_j$$

and directly by Proposition 5.1,

$$\vartheta_j \geqslant \lambda_j \tag{9.1}$$

We will show that on the event $\mathcal{E}_D$, there exists exactly one eigenvalue $\widetilde{\lambda}_j$ lying inside the disk

$$\mathtt{D}(\vartheta_j, \rho) := \{z \in \mathbb{C} : |z - \vartheta_j| \leqslant \rho\}$$

for $\rho \equiv \rho_j$ defined in (9.2). Denote

$$K(z) := \Lambda^{-1} - U^{\mathrm{T}}\Phi(z)U \quad \text{and} \quad B(z) := U^{\mathrm{T}}(G(z) - \Phi(z))U.$$

Set

$$g(z) := \det(K(z)) \quad \text{and} \quad f(z) = \det(\Lambda^{-1} - U^{\mathrm{T}}G(z)U) = \det(K(z) - B(z)).$$

By Lemma 21 from [73], the eigenvalues of $\widetilde{A}$ outside the $[-\|E\|, \|E\|]$ are the zeros of $f(z)$. Besides, the algebraic multiplicity of each eigenvalue matches the corresponding multiplicity of each zero. Next, we show that the zeros of $f(z)$ are close to the zeros of $g(z)$.

The following result is a special case of the generalized Rouché's theorem for operator functions (see [54]). We include a short proof for completeness.

LEMMA 9.1. *Let $\Gamma \subseteq \mathbb{C}$ be a simple closed contour. Assume $K(z)$ and $B(z)$ are analytic in a neighborhood of Γ and its interior. If*

$$\sup_{z \in \Gamma} \|K^{-1}(z)B(z)\| < 1,$$

then f and g have the same number of zeros inside Γ, counting multiplicity.

Proof. For $t \in [0, 1]$, define $F_t(z) := \det(K(z) - tB(z))$. For $z \in \Gamma$,

$$K(z) - tB(z) = K(z)(I - tK^{-1}(z)B(z)).$$

Since $\|tK^{-1}(z)B(z)\| < 1$, $I - tK^{-1}(z)B(z)$ is invertible. Hence, $F_t(z) \neq 0$ on Γ for all $t \in [0, 1]$. By the argument principle ([5, Section 5.2]), the number of zeros of F_t inside Γ is a constant in t. Taking $t = 0$ and $t = 1$ proves the claim. □

We will take a circle

$$\Gamma_j(\rho) := \{z \in \mathbb{C} : |z - \vartheta_j| = \rho\}$$

with the radius

$$\rho \equiv \rho_j := 10M + 15\frac{\mathrm{osc}(\mathcal{R})}{\lambda_j}. \tag{9.2}$$

To apply Lemma 9.1 on $\Gamma_j(\rho)$, we bound

$$\sup_{z\in\Gamma_j(\rho)} \|K^{-1}(z)B(z)\| \leqslant \sup_{z\in\Gamma_j(\rho)} \|K^{-1}(z)\| \cdot \sup_{z\in\Gamma_j(\rho)} \|B(z)\|.$$

To bound $\sup_{z\in\Gamma_j(\rho)} \|B(z)\|$, first note that $\Gamma_j(\rho) \subseteq \mathcal{S}_{\mathrm{out}}$. To see this, since $\lambda_j \geqslant 6\sqrt{R_{\max}}+100\rho_j$ and $\rho_j \leqslant \lambda_j/100$, for every $z \in \Gamma_j(\rho_j)$,

$$\begin{aligned}
|z| &\geqslant \vartheta_j - \rho_j \geqslant \lambda_j - \rho_j \geqslant 6\sqrt{R_{\max}},\\
|z| &\leqslant \vartheta_j + \rho_j \leqslant 1.16\lambda_j + \frac{\lambda_j}{100} \leqslant 2R_{\max}^3
\end{aligned}$$

Hence, on the event $\mathcal{E}_D$, we get

$$\sup_{z\in\Gamma_j(\rho)} \|B(z)\| \leqslant \left(\frac{100}{99}\right)^2 \frac{M}{\vartheta_j^2}. \tag{9.3}$$

To bound $\sup_{z\in\Gamma_j(\rho)} \|K^{-1}(z)\|$, we rewrite $K(z)$ for $z \in \Gamma_j(\rho)$ in block form. Without loss of generality, by relabeling the columns of U and the diagonal entries of Λ, we may assume that the index j is the first one. Namely, write

$$U = (u_j, U_{-j}) \quad \text{and} \quad \Lambda = \begin{pmatrix} \lambda_j & 0 \\ 0 & \Lambda_{-j} \end{pmatrix}.$$

Thus,

$$K(z) = \Lambda^{-1} - U^{\mathrm{T}}\Phi(z)U = \begin{pmatrix} \alpha_j(z) & -b_j(z)^{\mathrm{T}} \\ -b_j(z) & D_j(z) \end{pmatrix},$$

where

$$\alpha_j(z) := \lambda_j^{-1} - u_j^{\mathrm{T}}\Phi(z)u_j, \quad b_j(z) := U_{-j}^{\mathrm{T}}\Phi(z)u_j, \quad D_j(z) = \Lambda_{-j}^{-1} - U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}.$$

The following lemma provides the estimates of $|\alpha_j(z)|$, $\|b_j(z)\|$, and the smallest singular value $s_{\min}(D_j(z))$ on the circle $\Gamma_j(\rho)$. Its proof is deferred to Section 9.1.2.

LEMMA 9.2. *Fix $j \in [r_+]$. Assume $\lambda_j \geqslant 6\sqrt{R_{\max}} + 100\rho$ and λ_j is a simple eigenvalue of A. Assume the following gap condition holds:*

$$\min_{l<j}(\vartheta_l - \vartheta_j) \geqslant 50\rho \quad \textit{and} \quad \min_{l\in\mathcal{O}_+,\, l>j}(\vartheta_j - \vartheta_l) \geqslant 50\rho. \tag{9.4}$$

where $\mathcal{O}_+ := \{l \in [r_+] : \lambda_l \geqslant 6\sqrt{R_{\max}}\}$ *is the index set for the supercritical eigenvalues. Then for* $z \in \Gamma_j(\rho)$*, we have*

$$\frac{\rho}{4\vartheta_j^2} \leqslant |\alpha_j(z)| \leqslant \frac{3\rho}{\vartheta_j^2}, \tag{9.5}$$

$$\|b_j(z)\| \leqslant \frac{1}{10}\frac{\rho}{\vartheta_j^2}, \tag{9.6}$$

$$s_{\min}(D_j(z)) \geqslant 3\frac{\rho}{\vartheta_j^2}. \tag{9.7}$$

In particular, (9.7) suggests that $D_j(z)$ is invertible on $\Gamma_j(z)$ and

$$\|D_j^{-1}(z)\| \leqslant \frac{1}{3}\frac{\vartheta_j^2}{\rho}. \tag{9.8}$$

Define

$$h_j(z) := \alpha_j(z) - b_j(z)^{\mathrm{T}} D_j^{-1}(z) b_j(z)$$

for $z \in \Gamma_j(\rho)$. By Lemma 9.2, $h_j(z)$ is well-defined on $\Gamma_j(\rho)$. Besides,

$$|h_j(z)| \geqslant |\alpha_j(z)| - \|b_j(z)\|^2\|D_j^{-1}(z)\| \geqslant \frac{\rho}{4\vartheta_j^2} - \frac{1}{300}\frac{\rho}{\vartheta_j^2} = \frac{37}{150}\frac{\rho}{\vartheta_j^2} > 0$$

and

$$|h_j^{-1}(z)| \leqslant \frac{150}{37}\frac{\vartheta_j^2}{\rho}. \tag{9.9}$$

Now we are ready to estimate $\|K^{-1}(z)\|$ on $\Gamma_j(\rho)$. By Schur's complement,

$$K(z)^{-1} = \begin{pmatrix} h_j^{-1}(z) & h_j^{-1}(z) b_j(z)^{\mathrm{T}} D_j^{-1}(z) \\ h_j^{-1}(z) D_j^{-1}(z) b_j(z) & D_j^{-1}(z) + h_j^{-1}(z) D_j^{-1}(z) b_j(z) b_j(z)^{\mathrm{T}} D_j^{-1}(z) \end{pmatrix}$$

We decompose $K(z)^{-1}$ as

$$K(z)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & D_j^{-1}(z) \end{pmatrix} + h_j^{-1}(z) \begin{pmatrix} 1 & b_j(z)^{\mathrm{T}} D_j^{-1}(z) \\ D_j^{-1}(z) b_j(z) & D_j^{-1}(z) b_j(z) b_j(z)^{\mathrm{T}} D_j^{-1}(z) \end{pmatrix}.$$

Note that by setting $\mathtt{v} := b_j(z)^{\mathrm{T}} D_j^{-1}(z)$, we have

$$\mathtt{H} := \begin{pmatrix} I & b_j(z)^{\mathrm{T}} D_j^{-1}(z) \\ D_j^{-1}(z) b_j(z) & D_j^{-1}(z) b_j(z) b_j(z)^{\mathrm{T}} D_j^{-1}(z) \end{pmatrix} = \begin{pmatrix} 1 & \mathtt{v}^{\mathrm{T}} \\ \mathtt{v} & \mathtt{v}\mathtt{v}^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 1 \\ \mathtt{v} \end{pmatrix} \begin{pmatrix} 1 & \mathtt{v}^{\mathrm{T}} \end{pmatrix}.$$

Hence,

$$\|\mathtt{H}\| = 1 + \|\mathtt{v}\|^2 = 1 + \|b_j(z)^{\mathrm{T}} D_j^{-1}(z)\|^2 \leqslant 1 + \|b_j(z)\|^2 \cdot \|D_j^{-1}(z)\|^2.$$

It follows that

$$\begin{aligned} \|K(z)^{-1}\| &\leqslant \|D_j^{-1}(z)\| + |h_j^{-1}(z)| \cdot \|\mathtt{H}\| \\ &\leqslant \|D_j^{-1}(z)\| + |h_j^{-1}(z)| \left(1 + \|b_j(z)\|^2 \cdot \|D_j^{-1}(z)\|^2\right). \end{aligned} \tag{9.10}$$

Plugging the bounds in Lemma 9.2, (9.8), and (9.9), we get

$$\|K(z)^{-1}\| \leqslant \frac{1}{3}\frac{\vartheta_j^2}{\rho} + \frac{150}{37}\frac{\vartheta_j^2}{\rho}\left(1 + \frac{1}{900}\right) < \frac{9}{2}\frac{\vartheta_j^2}{\rho}.$$

Together with (9.3), we arrive at

$$\sup_{z\in\Gamma_j(\rho)} \|K^{-1}(z)B(z)\| \leqslant \sup_{z\in\Gamma_j(\rho)} \|K^{-1}(z)\| \cdot \sup_{z\in\Gamma_j(\rho)} \|B(z)\| \leqslant \frac{9}{2}\frac{\vartheta_j^2}{\rho} \cdot \left(\frac{100}{99}\right)^2 \frac{M}{\vartheta_j^2} < 1,$$

where we used $\rho \geqslant 5M$ in the last inequality.

By the assumption (9.4), $K(z)$ has exactly one root ϑ_j inside $\Gamma_j(\rho)$. By Lemma 9.1, there is exactly one eigenvalue $\widehat{\lambda}$ of $\widetilde{A}$ lying in the cycle $\Gamma_j(\rho)$. To complete the proof, it remains to show that $\widehat{\lambda} = \widetilde{\lambda}_j$.

To establish the index matching $\widehat{\lambda} = \widetilde{\lambda}_j$, it suffices to verify the following eigenvalue counts for the perturbed matrix $\widetilde{A}$:

1. There are exactly $j-1$ eigenvalues lie strictly to the right of $\Gamma_j(\rho)$;
2. There are exactly $n-j$ eigenvalues lie strictly to the left of $\Gamma_j(\rho)$.

As noted previously, the eigenvalues of $\widetilde{A}$ outside $[-\|E\|, \|E\|]$ are exactly the zeros of $f(z) = \det(K(z) - B(z))$, with matching algebraic multiplicities. To establish the above eigenvalue counts, we define new contours as follows. Recall that $\mathtt{D}(\vartheta_l, \rho) := \{z \in \mathbb{C} : |z - \vartheta_l| \leqslant \rho\}$ denotes the disk centered as ϑ_l with radius ρ. Set

$$D_j^+ := \bigcup_{l<j} \mathtt{D}(\vartheta_l, \rho), \qquad D_j^- := D_j^+ \cup \mathtt{D}(\vartheta_j, \rho), \tag{9.11}$$

and let $\mathcal{C}_j^\pm := \partial D_j^\pm$ be positively oriented on each connected component. By our gap assumption (9.4), the disk $\mathtt{D}(\vartheta_j, \rho)$ is disjoint from D_j^+, and every supercritical root excluded from $D_j^\pm$ lies with distance at least 49ρ from $\mathcal{C}_j^\pm$.

Define the auxiliary matrix

$$A_0(z) := \operatorname{diag}(\alpha_1(z), \ldots, \alpha_r(z)) \quad \text{with} \quad \alpha_l(z) := \lambda_l^{-1} - u_l^{\mathrm{T}}\Phi(z)u_l. \tag{9.12}$$

We show the next result whose proof is deferred to Section 9.1.1

LEMMA 9.3. *Under the assumption of Lemma 9.2, we have*

$$\sup_{z\in\mathcal{C}_j^\pm} \|A_0(z)^{-1}(K(z) - A_0(z))\| < 1, \qquad \sup_{z\in\mathcal{C}_j^\pm} \|K(z)^{-1}B(z)\| < 1.$$

Hence, by Lemma 9.1, $\det(A_0(z))$, $g(z) = \det(K(z))$, and $f(z) = \det(K(z) - B(z))$ have the same number of zeros inside each region $D_j^\pm$. Now

$$\det(A_0(z)) = \prod_{l=1}^{r} \alpha_l(z),$$

and among these factors on the right-hand side, only $\alpha_l(z)$ with $l \in \mathcal{O}_+$ have zeros in the supercritical regime, namely at $z = \vartheta_l$ (see Proposition 5.1). By construction, D_j^+ contains exactly the roots $\vartheta_1, \ldots, \vartheta_{j-1}$, while D_j^- contains exactly the roots $\vartheta_1, \ldots, \vartheta_j$. Therefore, $f(z)$ has exactly $j-1$ zeros in D_j^+ and exactly j zeros in D_j^-.

Since $D_j^- \backslash D_j^+ = \mathtt{D}(\vartheta_j, \rho)$ and $\Gamma_j(\rho) = \partial\, \mathtt{D}(\vartheta_j, \rho)$, it follows that $\det(K - B)$ has exactly one zero in $\mathtt{D}(\vartheta_j, \rho)$. This is precisely the eigenvalue $\widehat{\lambda}$ enclosed by $\Gamma_j(\rho)$. Therefore there are exactly $j-1$ eigenvalues of $\widetilde{A}$ strictly to the right of $\widehat{\lambda}$, and hence

$$\widehat{\lambda} = \widetilde{\lambda}_j.$$

9.1.1 Proof of Lemma 9.3

We estimate $\|A_0(z)^{-1}\|$, $\|K(z) - A_0(z)\|$, and $\|K^{-1}(z)\|$ respectively for $z \in \mathcal{C}_j^{\pm}$. Since

$$K(z) - A_0(z) = -\mathrm{Off}(U^{\mathrm{T}} \Phi(z) U),$$

we have

$$\|K(z) - A_0(z)\| \leqslant \|\mathrm{Off}(U^{\mathrm{T}} \Phi(z) U)\|.$$

Claim. For $z \in \mathcal{C}_j^{\pm}$, the following estimates hold:

$$|\alpha_l(z)| \geqslant \frac{1}{2} \frac{\rho}{|z|^2}, \tag{9.13}$$

$$\|b_j(z)\| \leqslant \frac{2}{25} \frac{\rho}{|z|^2}, \tag{9.14}$$

$$\|\mathrm{Off}(U^{\mathrm{T}} \Phi(z) U)\| \leqslant \frac{1}{5} \frac{\rho}{|z|^2}, \tag{9.15}$$

$$\|\mathrm{Off}(U^{\mathrm{T}} \varepsilon(z) U)\| \leqslant \frac{1}{10} \frac{\rho}{|z|^2}. \tag{9.16}$$

We defer the proof of this claim to the end of this section. Using these bounds, we first have

$$\|A_0(z)^{-1}\| = \max_{l \in [r]} |\alpha_l^{-1}(z) = \frac{1}{|\min_{l \in [r]} \alpha_l(z)|} \leqslant 2 \frac{|z|^2}{\rho}$$

Hence,

$$\begin{aligned} \sup_{z \in \mathcal{C}_j^{\pm}} \|A_0(z)^{-1}(K(z) - A_0(z))\| &\leqslant \sup_{z \in \mathcal{C}_j^{\pm}} \|A_0(z)^{-1}\| \sup_{z \in \mathcal{C}_j^{\pm}} \|K(z) - A_0(z))\| \\ &\leqslant 2 \frac{|z|^2}{\rho} \cdot \frac{1}{5} \frac{\rho}{|z|^2} < 1. \end{aligned}$$

Next, we estimate $\|K(z)^{-1}\|$ for $z \in \mathcal{C}_j^{\pm}$, which is similar to the previous estimates on $\Gamma_j(\rho)$. Since $\left\|\mathrm{Off}\left(U_{-j}^{\mathrm{T}} \Phi(z) U_{-j}\right)\right\| \leqslant \left\|\mathrm{Off}\left(U^{\mathrm{T}} \Phi(z) U\right)\right\|$, from (9.22) and by the **Claim**, we get

$$\begin{aligned} s_{\min}(D_j(z)) &\geqslant \min_{l \in [r] \backslash \{j\}} |\alpha_l(z)| - \left\|\mathrm{Off}\left(U_{-j}^{\mathrm{T}} \Phi(z) U_{-j}\right)\right\| \\ &\geqslant \frac{1}{2} \frac{\rho}{|z|^2} - \frac{1}{5} \frac{\rho}{|z|^2} = \frac{3}{10} \frac{\rho}{|z|^2}. \end{aligned}$$

and thus

$$\|D_j^{-1}(z)\| \leqslant \frac{10}{3}\frac{|z|^2}{\rho}.$$

Recall that $h_j(z) = \alpha_j(z) - b_j(z)^{\mathrm{T}} D_j^{-1}(z) b_j(z)$. By the claim,

$$|h_j(z)| \geqslant |\alpha_j(z)| - \|b_j(z)\|^2 \|D_j^{-1}(z)\| \geqslant \frac{1}{2}\frac{\rho}{|z|^2} - \left(\frac{2}{25}\frac{\rho}{|z|^2}\right)^2 \frac{3}{10}\frac{|z|^2}{\rho} = \frac{359}{750}\frac{\rho}{|z|^2}.$$

Hence, $|h_j^{-1}(z)| \leqslant \frac{750}{359}\frac{|z|^2}{\rho}$ and by (9.10),

$$\begin{aligned}
\|K(z)^{-1}\| &\leqslant \|D_j^{-1}(z)\| + |h_j^{-1}(z)| \left(1 + \|b_j(z)\|^2 \cdot \|D_j^{-1}(z)\|^2\right) \\
&\leqslant \frac{10}{3}\frac{|z|^2}{\rho} + \frac{750}{359}\frac{|z|^2}{\rho}\left(1 + \left(\frac{2}{25}\frac{\rho}{|z|^2}\right)^2 \left(\frac{3}{10}\frac{|z|^2}{\rho}\right)^2\right) \\
&< 5.5\frac{|z|^2}{\rho}.
\end{aligned}$$

Finally,

$$\sup_{z\in\mathcal{C}_j^{\pm}} \|K^{-1}(z)B(z)\| \leqslant \sup_{z\in\mathcal{C}_j^{\pm}} \|K^{-1}(z)\| \cdot \sup_{z\in\mathcal{C}_j^{\pm}} \|B(z)\| \leqslant 5.5\frac{|z|^2}{\rho}\frac{M}{|z|^2} < 1,$$

where we used $\rho \geqslant 10M$ in the last inequality.

To finish the proof, it remains to prove the claim.

Proof of (9.13) We bound $|\alpha_l(z)|$ for $z \in \mathcal{C}_j^{\pm}$. The proof is similar to that of (9.29). We split the discussion into three cases: $l \in \mathcal{O}_+$, $l \in [r_+]\backslash\mathcal{O}_+$, and $l \in [r]\backslash[r_+]$.

Case 1: $l \in \mathcal{O}_+$. We have $\lambda_l \geqslant 6\sqrt{R_{\max}}$. By Proposition 5.1, define ϑ_l as the unique solution to $\alpha_l(z) = 0$. Also, $\vartheta_l \geqslant 5\sqrt{R_{\max}}$.

For $z = x + \sqrt{-1}y \in \mathcal{C}_j^{\pm}$, we have $|y| \leqslant \rho$ and $x \geqslant \vartheta_j - \rho \geqslant \frac{99}{100}\vartheta_j > 4\sqrt{R_{\max}}$. Note that using the same estimates as (9.20) and (9.21), we have

$$\frac{4}{5}\frac{1}{x^2} \leqslant \alpha_l'(x) = -u_l^{\mathrm{T}}\Phi'(x)u_l \leqslant \frac{3}{x^2}. \tag{9.17}$$

Indeed, we refined the constant for the lower bound using $x \geqslant 4\sqrt{R_{\max}}$:

$$\alpha_l'(x) = -u_l^{\mathrm{T}}\Phi'(x)u_l \geqslant \frac{1}{x^2} - \frac{243}{5}\frac{R_{\max}^2}{x^6} \geqslant \frac{1}{x^2}\left(1 - \frac{243}{5\cdot 256}\right) > \frac{4}{5x^2}.$$

Since $\alpha_l(\vartheta_l) = 0$, from $|\alpha_l(x)| = |\int_{\vartheta_l}^x \alpha_l'(t)\,dt|$, we get

$$\frac{4}{5}\frac{|x-\vartheta_l|}{x\vartheta_l} \leqslant |\alpha_l(x)| \leqslant 3\frac{|x-\vartheta_l|}{x\vartheta_l}. \tag{9.18}$$

Now the Taylor expansion of $\alpha_l(z)$ at x yields:

$$\alpha_l(z) = \alpha_l(x) + \sqrt{-1}y\alpha_l'(x) + \frac{1}{2}\alpha_l''(\zeta)(\sqrt{-1}y)^2,$$

where ζ lies in a vertical segment connecting x and z. In particular, $|\zeta| \geqslant x \geqslant 4\sqrt{R_{\max}}$. It follows from part (i) of Lemma 5.1 that

$$|\alpha_l''(\zeta)| = |u_l^{\mathrm{T}}\Phi''(\zeta)u_l| \leqslant \|\Phi''(\zeta)\| \leqslant \frac{28}{|\zeta|^3} \leqslant \frac{28}{x^3}.$$

Next, plugging in (9.17) and (9.18) yields

$$\begin{aligned} |\alpha_l(x) + \sqrt{-1}y\alpha_l'(x)| &= \sqrt{\alpha_l(x)^2 + y^2\alpha_l'(x)^2} \\ &\geqslant \sqrt{\frac{16}{25}\frac{(x-\vartheta_l)^2}{x^2\vartheta_l^2} + \frac{16}{25}\frac{y^2}{x^4}} = \frac{4}{5}\frac{1}{x^2}\sqrt{(x-\vartheta_l)^2\frac{x^2}{\vartheta_l^2} + y^2}. \end{aligned}$$

If $\vartheta_l \leqslant \frac{100}{99}x$, then $x/\vartheta_l \geqslant 0.99$ and

$$|\alpha_l(x) + \sqrt{-1}y\alpha_l'(x)| \geqslant \frac{4}{5x^2}0.99\sqrt{(x-\vartheta_l)^2 + y^2} = 0.792\frac{|z-\vartheta_l|}{x^2} \geqslant 0.792\frac{\rho}{x^2}.$$

If $\vartheta_l > \frac{100}{99}x$, we use the fact that the function $f(t) = \frac{t-x}{xt}$ is increasing for $t > x$. Its minimum value occurs at the boundary $t = \frac{100}{99}x$. By (9.18),

$$|\alpha_l(x) + \sqrt{-1}y\alpha_l'(x)| \geqslant |\alpha_l(x)| \geqslant \frac{4}{5}\frac{\vartheta_l - x}{x\vartheta_l} \geqslant \frac{4}{5}\frac{\frac{100}{99}x - x}{x\cdot\frac{100}{99}x} = \frac{4x}{500x^2}.$$

Since $x \geqslant 99\rho$, we have $\frac{4x}{500x^2} \geqslant \frac{4\cdot 99\rho}{500x^2} = 0.792\frac{\rho}{x^2}$.

Hence, we obtain

$$\begin{aligned} |\alpha_l(z)| &\geqslant |\alpha_l(x) + \sqrt{-1}y\alpha_l'(x)| - 14\frac{y^2}{x^3} \\ &\geqslant 0.792\frac{\rho}{x^2} - \frac{14}{99}\frac{\rho}{x^2} > \frac{\rho}{2x^2} \geqslant \frac{\rho}{2|z|^2}, \end{aligned}$$

where we used $|z| \geqslant x \geqslant 99\rho \geqslant 99|y|$.

Case 2: $l \in [r_+]\backslash\mathcal{O}_+$. The lower bound on $|\alpha_l(z)|$ follows the same approach as (9.27). We skip the proof. In this case, we have

$$|\alpha_l(z)| \geqslant 70\frac{\rho}{\vartheta_j^2} \geqslant 70\left(\frac{99}{100}\right)^2\frac{\rho}{|z|^2} > 60\frac{\rho}{|z|^2}.$$

Case 3: $l \in [r]\backslash[r_+]$. We have $\lambda_l < 0$. For $z = x + \sqrt{-1}y \in \mathcal{C}_j^{\pm}$, we have $x \geqslant \vartheta_j - \rho \geqslant \frac{99}{100}\vartheta_j 4\sqrt{R_{\max}}$. From (5.3), we expand

$$\alpha_l(z) = \frac{1}{\lambda_l} - u_l^{\mathrm{T}}\Phi(z)u_l = \frac{1}{\lambda_l} - \frac{1}{z} - \frac{1}{z^3}u_l^{\mathrm{T}}\mathcal{R}u_l - u_l^{\mathrm{T}}\varepsilon(z)u_l.$$

Note that

$$\left|\frac{1}{\lambda_l}-\frac{1}{z}\right| \geqslant \left|\mathrm{Re}\left(\frac{1}{\lambda_l}-\frac{1}{z}\right)\right| = \frac{1}{|\lambda_l|}+\frac{x}{|z|^2} > \frac{x}{|z|^2}.$$

Furthermore, we have

$$\left|\frac{1}{z^3}u_l^{\mathrm{T}}\mathcal{R}u_l + u_l^{\mathrm{T}}\varepsilon(z)u_l\right| \leqslant \frac{R_{\max}}{|z|^3}+\frac{45}{8}\frac{R_{\max}^2}{|z|^5} \leqslant \frac{173}{128}\frac{R_{\max}}{|z|^3} \leqslant \frac{173}{128\cdot 16}\frac{x}{|z|^2}$$

using the bound $|z| \geqslant x \geqslant 4\sqrt{R_{\max}}$. Hence,

$$|\alpha_l(z)| \geqslant \frac{x}{|z|^2} - \frac{173}{128\cdot 16}\frac{x}{|z|^2} > 90\frac{\rho}{|z|^2}.$$

Combining the above estimates, we have proved (9.13).

Proof of (9.14) The proof is almost identical to that of (9.6). We briefly it here. Following the same approach as (9.6), we first get

$$\|b_j(z)\| \leqslant \frac{\mathrm{osc}(\mathcal{R})}{2|z|^3} + \frac{15R_{\max}}{2|z|^5}\mathrm{osc}(\mathcal{R}) < \frac{\mathrm{osc}(\mathcal{R})}{|z|^3}.$$

Since $|z| \geqslant \vartheta_j - \rho \geqslant \frac{99}{100}\vartheta_j$ and

$$\rho \geqslant 13\frac{\mathrm{osc}(\mathcal{R})}{\vartheta_j},$$

we further get

$$\frac{\mathrm{osc}(\mathcal{R})}{|z|^3} \leqslant \frac{100}{99}\frac{\mathrm{osc}(\mathcal{R})}{\vartheta_j}\frac{1}{|z|^2} \leqslant \frac{100}{99\cdot 13}\frac{\rho}{|z|^2}, \tag{9.19}$$

and hence

$$\|b_j(z)\| \leqslant \frac{2}{25}\frac{\rho}{|z|^2}.$$

Proofs of (9.15) **and** (9.16) The bound for $\mathrm{Off}\big(U^{\mathrm{T}}\Phi(z)U\big)$ are almost identical to the bound for $\mathrm{Off}\big(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\big)$ derived in (9.7). We briefly sketch it here. It follows from the same steps that

$$\|\mathrm{Off}(U_{-j}^{\mathrm{T}}\varepsilon(z)U_{-j})\| \leqslant \frac{15R_{\max}}{\sqrt{2}|z|^5}\mathrm{osc}(\mathcal{R}) \leqslant \frac{15}{9\sqrt{2}}\frac{\mathrm{osc}(\mathcal{R})}{|z|^3} \leqslant \frac{1}{10}\frac{\rho}{|z|^2}$$

and

$$\begin{aligned}\big\|\mathrm{Off}\big(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\big)\big\| &\leqslant \frac{\mathrm{osc}(\mathcal{R})}{2|z|^3} + \frac{15R_{\max}}{\sqrt{2}|z|^5}\mathrm{osc}(\mathcal{R}) \\ &\leqslant \frac{\mathrm{osc}(\mathcal{R})}{|z|^3}\left(\frac{1}{2}+\frac{15}{9\sqrt{2}}\right) \leqslant \frac{1}{5}\frac{\rho}{|z|^2}\end{aligned}$$

where we used (9.19).

9.1.2 Proof of Lemma 9.2

Recall from (9.2) that

$$\rho = 5M + 15\frac{\operatorname{osc}(\mathcal{R})}{\lambda_j}.$$

By our assumption, $\rho \leqslant \lambda_j/100$. From the estimates in (9.1), we also have

$$\rho \geqslant 15 \cdot 0.87\frac{\operatorname{osc}(\mathcal{R})}{\vartheta_j} > 13\frac{\operatorname{osc}(\mathcal{R})}{\vartheta_j}.$$

For $z \in \Gamma_j(\rho)$ satisfying $|z - \vartheta_j| = \rho$, we have

$$\frac{99}{100}\vartheta_j \leqslant \vartheta_j - \rho \leqslant |z| \leqslant \vartheta_j - \rho \leqslant \frac{101}{100}\vartheta_j.$$

Proof of (9.5). We first estimate $|\alpha_j(z)|$. Recall that $\alpha_j(\vartheta_j) = 0$. By Taylor's theorem,

$$\alpha_j(z) = \alpha_j'(\vartheta_j)(z - \vartheta_j) + \mathtt{E}_2,$$

where

$$|\mathtt{E}_2| \leqslant \frac{1}{2}|z - \vartheta_j|^2 \max_{|\zeta - \vartheta_j| \leqslant \rho} |\alpha_j''(\zeta)| = \frac{1}{2}\rho^2 \max_{|\zeta - \vartheta_j| \leqslant \rho} |\alpha_j''(\zeta)|.$$

By part (ii) of Lemma 5.1 (see also (A.15)),

$$\alpha_j'(\vartheta_j) = -u_j^{\mathrm{T}}\Phi'(\vartheta_j)u_j \geqslant \frac{1}{\vartheta_j^2} - \frac{243}{5}\frac{R_{\max}^2}{\vartheta_j^6} \geqslant \frac{1}{\vartheta_j^2}\left(1 - \frac{243}{5 \cdot 81}\right) = \frac{2}{5\vartheta_j^2}. \tag{9.20}$$

For ζ satisfying $|\zeta - \vartheta_j| \leqslant \rho$, we have $|\zeta| \geqslant \vartheta_j - \rho \geqslant \frac{99}{100}\vartheta_j > 4\sqrt{R_{\max}}$. It follows from part (i) of Lemma 5.1 that

$$|\alpha_j''(\zeta)| = |u_j^{\mathrm{T}}\Phi''(\zeta)u_j| \leqslant \|\Phi''(\zeta)\| \leqslant \frac{28}{|\zeta|^3} \leqslant 28\left(\frac{100}{99}\right)^3\frac{1}{\vartheta_j^3} < \frac{29}{\vartheta_j^3}.$$

Thus,

$$|\mathtt{E}_2| \leqslant \frac{29}{2}\frac{\rho^2}{\vartheta_j^3} \leqslant \frac{29}{200}\frac{\rho}{\vartheta_j^2}$$

and

$$|\alpha_j(z)| \geqslant \frac{2\rho}{5\vartheta_j^2} - \frac{29}{200}\frac{\rho}{\vartheta_j^2} > \frac{\rho}{4\vartheta_j^2}.$$

For the upper bound, observe that

$$|\alpha_j(z)| = |\alpha_j(z) - \alpha_j(\vartheta_j)| \leqslant \max_{|\zeta - \vartheta_j| \leqslant \rho} |\alpha_j'(\zeta)| \cdot \rho.$$

By part (ii) of Lemma 5.1 (see also (A.15)), we have

$$|\alpha_j'(\zeta)| \leqslant \frac{1}{|\zeta|^2} + \frac{3\mathcal{V}_{jj}}{|\zeta|^4} + \frac{243}{5}\frac{R_{\max}^2}{|\zeta|^6} \leqslant \frac{2}{|\zeta|^2} \leqslant \frac{3}{\vartheta_j^2} \tag{9.21}$$

using $|\zeta| \geqslant \frac{99}{100}\vartheta_j$ and $\vartheta_j > 5\sqrt{R_{\max}}$. Hence,

$$|\alpha_j(z)| \leqslant \frac{3\rho}{\vartheta_j^2}.$$

Proof of (9.6). Next, we estimate $\|b_j(z)\| = \|U_{-j}^{\mathrm{T}}\Phi(z)u_j\|$. Since $U_{-j}^{\mathrm{T}}u_j = 0$, using the expansion of $\Phi(z)$ in (5.3), we immediately get

$$b_j(z) = \frac{1}{z^3}U_{-j}^{\mathrm{T}}\mathcal{R}u_j + U_{-j}^{\mathrm{T}}\varepsilon(z)u_j.$$

Thus,

$$\|b_j(z)\| \leqslant \frac{\|U_{-j}^{\mathrm{T}}\mathcal{R}u_j\|}{|z|^3} + \|U_{-j}^{\mathrm{T}}\varepsilon(z)u_j\|.$$

Since $U_{-j}^{\mathrm{T}}u_j = 0$,

$$\|U_{-j}^{\mathrm{T}}\mathcal{R}u_j\| = \|U_{-j}^{\mathrm{T}}(\mathcal{R} - cI)u_j\| \leqslant \|\mathcal{R} - cI\| \leqslant \frac{1}{2}\mathrm{osc}(\mathcal{R})$$

by selecting $c = \frac{1}{2}(R_{\max} + R_{\min})$. Similarly, we also have

$$\|U_{-j}^{\mathrm{T}}\varepsilon(z)u_j\| \leqslant \frac{1}{2}\mathrm{osc}(\varepsilon(z)) \leqslant \frac{15R_{\max}}{2|z|^5}\mathrm{osc}(\mathcal{R}),$$

where the last inequality follows from Lemma 5.2. Thus

$$\|b_j(z)\| \leqslant \frac{\mathrm{osc}(\mathcal{R})}{2|z|^3} + \frac{15R_{\max}}{2|z|^5}\mathrm{osc}(\mathcal{R}) < \frac{\mathrm{osc}(\mathcal{R})}{|z|^3} \leqslant \left(\frac{100}{99}\right)^2\frac{\mathrm{osc}(\mathcal{R})}{\vartheta_j^3} < \frac{1}{10}\frac{\rho}{\vartheta_j^2}$$

by our assumption $\rho \geqslant 15\frac{\mathrm{osc}(\mathcal{R})}{\lambda_j}$.

Proof of (9.7). Finally, we show the lower bound on $s_{\min}(D_j(z))$. For each $l \in [r]$, set

$$\alpha_l(z) := \lambda_l^{-1} - u_l^{\mathrm{T}}\Phi(z)u_l.$$

We rewrite

$$\begin{aligned} D_j(z) &= \Lambda_{-j}^{-1} - U_{-j}^{\mathrm{T}}\Phi(z)U_{-j} \\ &= \Lambda_{-j}^{-1} - \mathrm{diag}(u_l^{\mathrm{T}}\Phi(z)u_l)_{l\in[r]\setminus\{j\}} - \mathrm{Off}\big(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\big) \\ &= \mathrm{diag}(\alpha_l(z))_{l\in[r]\setminus\{j\}} - \mathrm{Off}\big(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\big). \end{aligned}$$

Thus,

$$s_{\min}(D_j(z)) \geqslant \min_{l\in[r]\setminus\{j\}} |\alpha_l(z)| - \Big\|\mathrm{Off}\big(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\big)\Big\|. \tag{9.22}$$

We first bound $\mathrm{Off}\big(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\big)$. By (5.3),

$$\begin{aligned} \mathrm{Off}\big(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\big) &= \mathrm{Off}\big(\frac{1}{z}I + \frac{1}{z^3}U_{-j}^{\mathrm{T}}\mathcal{R}U_{-j} + U_{-j}^{\mathrm{T}}\varepsilon(z)U_{-j}\big) \\ &= \mathrm{Off}\big(\frac{1}{z^3}U_{-j}^{\mathrm{T}}\mathcal{R}U_{-j}\big) + \mathrm{Off}\big(U_{-j}^{\mathrm{T}}\varepsilon(z)U_{-j}\big). \end{aligned} \tag{9.23}$$

Note that

$$\mathrm{Off}(U_{-j}^{\mathrm{T}}(\mathcal{R}-c_1 I)U_{-j}) = \mathrm{Off}(U_{-j}^{\mathrm{T}}\mathcal{R}U_{-j}).$$

Choosing $c_1 = \frac{1}{2}(R_{\max}+R_{\min})$, we get

$$\left\|\mathrm{Off}\left(U_{-j}^{\mathrm{T}}\mathcal{R}U_{-j}\right)\right\| \leqslant \|\mathcal{R}-c_1 I\| \leqslant \frac{1}{2}\mathrm{osc}(\mathcal{R}).$$

Similarly, we also have

$$\|\mathrm{Off}(U_{-j}^{\mathrm{T}}\varepsilon(z)U_{-j})\| = \|\mathrm{Off}(U_{-j}^{\mathrm{T}}(\varepsilon(z)-c_2 I)U_{-j})\| \leqslant \|\varepsilon(z)-c_2 I\|.$$

Decompose $\varepsilon(z) = \mathrm{diag}(\varepsilon_k(z)) = \mathrm{diag}(x_k) + \sqrt{-1}\mathrm{diag}(y_k)$. Set $c_2 = c_x + \sqrt{-1}c_y = \frac{1}{2}(\max_k x_k + \min_k x_k) + \sqrt{-1}(\max_k y_k + \min_k y_k)$. Then

$$|\varepsilon_k(z)-c_2|^2 = |x_k - c_x|^2 + |y_k - c_y|^2| \leqslant \frac{1}{4}\mathrm{osc}(x)^2 + \frac{1}{4}\mathrm{osc}(y)^2 \leqslant \frac{1}{2}\mathrm{osc}(\varepsilon(z))^2.$$

It follows that

$$\|\mathrm{Off}(U_{-j}^{\mathrm{T}}\varepsilon(z)U_{-j})\| \leqslant \frac{1}{\sqrt{2}}\mathrm{osc}(\varepsilon(z)) \leqslant \frac{15R_{\max}}{\sqrt{2}|z|^5}\mathrm{osc}(\mathcal{R}).$$

The last inequality is by Lemma 5.2. Combining these estimates with (9.23) yields

$$\begin{aligned}\left\|\mathrm{Off}\left(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\right)\right\| &\leqslant \frac{\mathrm{osc}(\mathcal{R})}{2|z|^3} + \frac{15R_{\max}}{\sqrt{2}|z|^5}\mathrm{osc}(\mathcal{R}) \\ &\leqslant \frac{\mathrm{osc}(\mathcal{R})}{|z|^3}\left(\frac{1}{2}+\frac{15}{9\sqrt{2}}\right) \\ &< 1.8\frac{\mathrm{osc}(\mathcal{R})}{\vartheta_j^3} \leqslant \frac{1.8}{13}\frac{\rho}{\vartheta_j^2}\end{aligned} \tag{9.24}$$

where we used $|z| \geqslant \frac{99}{100}\vartheta_j$ and $\rho > 13\frac{\mathrm{osc}(\mathcal{R})}{\vartheta_j}$.

It remains to bound $\min_{l\in[r]\setminus\{j\}} |\alpha_l(z)|$ where

$$\alpha_l(z) = \lambda_l^{-1} - u_l^{\mathrm{T}}\Phi(z)u_l.$$

Define

$$\mathcal{O}_+ := \{l \in [r_+] : \lambda_l \geqslant 6\sqrt{R_{\max}}\}.$$

We split the discussion into three cases: $l \in \mathcal{O}_+$, $l \in [r_+]\setminus\mathcal{O}_+$ and $l \in [r]\setminus[r_+]$.

Case 1: $l \in \mathcal{O}_+$. We have $\lambda_l \geqslant 6\sqrt{R_{\max}}$. By Proposition 5.1, define ϑ_l as the unique solution to $\alpha_l(z) = 0$. Also, $\vartheta_l \geqslant 5\sqrt{R_{\max}}$. Using similar estimates as (9.20), we get, for any real $t \geqslant 3\sqrt{R_{\max}}$,

$$\alpha_l'(t) = -u_l^{\mathrm{T}}\Phi'(t)u_l \geqslant \frac{2}{5t^2}.$$

Since $\alpha_l(\vartheta_l) = 0$,

$$|\alpha_l(\vartheta_j)| = \left|\int_{\vartheta_l}^{\vartheta_j}\alpha_l'(t)\,dt\right| \geqslant \left|\int_{\vartheta_l}^{\vartheta_j}\frac{2}{5t^2}\,dt\right| = \frac{2}{5}\frac{|\vartheta_l-\vartheta_j|}{\vartheta_l\vartheta_j} \geqslant \frac{2}{5}\frac{|\vartheta_l-\vartheta_j|}{\vartheta_j(\vartheta_j+|\vartheta_l-\vartheta_j|)}.$$

The last step is from $\vartheta_l \leqslant \vartheta_j + |\vartheta_l - \vartheta_j|$. Since $|\vartheta_l - \vartheta_j| \geqslant 50\rho$ by our assumption (9.4), together with $\rho \leqslant \vartheta_j/100$, we get

$$|\alpha_l(\vartheta_j)| \geqslant \frac{40}{3}\frac{\rho}{\vartheta_j^2}.$$

Next, using the same estimates as in (9.21), we also have

$$|\alpha_l(\vartheta_j) - \alpha_l(z)| \leqslant \max_{|\zeta-\vartheta_j|\leqslant\rho} |\alpha_l'(\zeta)| \cdot \rho \leqslant 3\frac{\rho}{\vartheta_j^3}.$$

Hence,

$$|\alpha_l(z)| \geqslant |\alpha_l(\vartheta_j)| - |\alpha_l(\vartheta_j) - \alpha_l(z)| \geqslant \left(\frac{40}{3} - 3\right)\frac{\rho}{\vartheta_j^2} = \frac{10}{3}\frac{\rho}{\vartheta_j^2}. \tag{9.25}$$

Case 2: $l \in [r_+]\backslash\mathcal{O}_+$. We have $0 < \lambda_l < 6\sqrt{R_{\max}} \leqslant \lambda_j$. From

$$\alpha_l(z) - \alpha_j(z) = \left(\lambda_l^{-1} - \lambda_j^{-1}\right) - \left(u_l^{\mathrm{T}}\Phi(z)u_l - u_j^{\mathrm{T}}\Phi(z)u_j\right),$$

observe that

$$|\alpha_l(z) - \alpha_j(z)| \geqslant \frac{\lambda_j - \lambda_l}{\lambda_j\lambda_l} - \left|u_l^{\mathrm{T}}\Phi(z)u_l - u_j^{\mathrm{T}}\Phi(z)u_j\right|. \tag{9.26}$$

Furthermore, by (5.3),

$$\begin{aligned}\left|u_l^{\mathrm{T}}\Phi(z)u_l - u_j^{\mathrm{T}}\Phi(z)u_j\right| &= \left|\frac{1}{z^3}u_l^{\mathrm{T}}\mathcal{R}u_l - \frac{1}{z^3}u_j^{\mathrm{T}}\mathcal{R}u_j - (u_l^{\mathrm{T}}\varepsilon(z)u_l - u_j^{\mathrm{T}}\varepsilon(z)u_j)\right| \\ &\leqslant \frac{1}{|z|^3}|u_l^{\mathrm{T}}\mathcal{R}u_l - u_j^{\mathrm{T}}\mathcal{R}u_j| + |u_l^{\mathrm{T}}\varepsilon(z)u_l - u_j^{\mathrm{T}}\varepsilon(z)u_j| \\ &\leqslant \frac{\mathrm{osc}(\mathcal{R})}{|z|^3} + \mathrm{osc}(\varepsilon(z)).\end{aligned}$$

For the last inequality above, since u_l, u_j are unit vectors, their quadratic forms represent convex combinations of the diagonal entries. Therefore, the difference between them is strictly bounded by the maximum pairwise distance in the complex plane, which is the oscillation.

By Lemma 5.2,

$$\left|u_l^{\mathrm{T}}\Phi(z)u_l - u_j^{\mathrm{T}}\Phi(z)u_j\right| \leqslant \frac{\mathrm{osc}(\mathcal{R})}{|z|^3}\left(1 + \frac{15R_{\max}}{|z|^2}\right) \leqslant \frac{8}{3}\frac{\mathrm{osc}(\mathcal{R})}{|z|^3}.$$

Continuing from (9.26), together with $|\alpha_j(z)| \leqslant \frac{3\rho}{\vartheta_j^2}$ from (9.5) and the triangle inequality, we obtain

$$|\alpha_l(z)| \geqslant |\alpha_l(z) - \alpha_j(z)| - |\alpha_j(z)| \geqslant \frac{\lambda_j - \lambda_l}{\lambda_j^2} - \frac{8}{3}\frac{\mathrm{osc}(\mathcal{R})}{|z|^3} - \frac{3\rho}{\vartheta_j^2}.$$

Since $\lambda_l < 6\sqrt{R_{\max}}$ and $\lambda_j \geqslant 6\sqrt{R_{\max}} + 100\rho$, we have $\lambda_j - \lambda_l \geqslant 100\rho$. Using our assumptions, $|z| \geqslant \frac{99}{100}\vartheta_j$, $0.87\lambda_j \leqslant \vartheta_j \leqslant 1.16\lambda_j$ from (9.1), and $\rho > 13\frac{\operatorname{osc}(\mathcal{R})}{\vartheta_j}$. Thus we get

$$|\alpha_l(z)| \geqslant (100 \cdot 0.87^2 - 3)\frac{\rho}{\vartheta_j^2} - \frac{8}{3}\left(\frac{100}{99}\right)^3 \frac{\operatorname{osc}(\mathcal{R})}{\vartheta_j^3} > 70\frac{\rho}{\vartheta_j^2}. \tag{9.27}$$

Case 3: $l \in [r]\backslash[r_+]$. We have $\lambda_l < 0$. Using the expansion (5.3), we get

$$\begin{aligned} u_l^{\mathrm{T}}\Phi(\vartheta_j)u_l &= \frac{1}{\vartheta_j} + \frac{\mathcal{V}_{ll}}{\vartheta_j^3} + \epsilon_l(\vartheta_j) \\ &\geqslant \frac{1}{\vartheta_j} - \|\epsilon_l(\vartheta_j)\| \geqslant \frac{1}{\vartheta_j} - \frac{45}{8}\frac{R_{\max}^2}{\vartheta_j^5} \\ &\geqslant \frac{1}{\vartheta_j} - \frac{5}{72}\frac{1}{\vartheta_j} = \frac{67}{72}\frac{1}{\vartheta_j}. \end{aligned}$$

Thus, $\alpha_l(\vartheta_j) = \lambda_l^{-1} - u_l^{\mathrm{T}}\Phi(\vartheta_j)u_l < 0$ and

$$|\alpha_l(\vartheta_j)| \geqslant \frac{67}{72}\frac{1}{\vartheta_j} \geqslant \frac{6700}{72}\frac{\rho}{\vartheta_j^2}.$$

Using similar estimates as (9.20), we have $|\alpha_j'(\zeta)| \leqslant \frac{3}{\vartheta_j^2}$ for any ζ satisfying $|\zeta - \vartheta_j| \leqslant \rho$. Consequently,

$$|\alpha_l(z) - \alpha_l(\vartheta_j)| \leqslant \rho \cdot \max_{|\zeta - \vartheta_j| \leqslant \rho} |\alpha_j'(\zeta)| \leqslant 3\frac{\rho}{\vartheta_j^2}$$

and

$$|\alpha_l(z)| \geqslant |\alpha_l(\vartheta_j)| - |\alpha_l(z) - \alpha_l(\vartheta_j)| \geqslant \left(\frac{6700}{72} - 3\right)\frac{\rho}{\vartheta_j^2} > 30\frac{\rho}{\vartheta_j^2}. \tag{9.28}$$

Finally, combining (9.25), (9.27), and (9.28), we obtain

$$\min_{l \in [r]\backslash\{j\}} |\alpha_l(z)| \geqslant \frac{10}{3}\frac{\rho}{\vartheta_j^2}. \tag{9.29}$$

Continuing from (9.22), together with (9.24), we arrive at

$$s_{\min}(D_j(z)) \geqslant \min_{l \in [r]\backslash\{j\}} |\alpha_l(z)| - \left\|\operatorname{Off}\left(U_{-j}^{\mathrm{T}}\Phi(z)U_{-j}\right)\right\| \geqslant \left(\frac{10}{3} - \frac{1.8}{13}\right)\frac{\rho}{\vartheta_j^2} > 3\frac{\rho}{\vartheta_j^2}.$$

This completes the proof.

9.2 Proof of Theorem 6

Theorem 6 will be proved using similar techniques and estimates as Theorem 5. We briefly sketch it here. Recall that $\rho \equiv \rho_j = 10M + 15\frac{\mathrm{osc}(\mathcal{R})}{\lambda_j}$.

To establish the lower bound for the top-j outlier cluster, we define the block contour enclosing the deterministic roots $\vartheta_1, \ldots, \vartheta_j$. Following our previous notation (9.11), let

$$D_j^- := \bigcup_{l=1}^{j} \mathtt{D}(\vartheta_l, \rho_j),$$

and let $\mathcal{C}_j^- := \partial D_j^-$ be its positively oriented boundary. By our gap assumption $\min_{l>k}(\underline{\vartheta}_k - \vartheta_l) \geqslant 50\rho_k$, every supercritical root excluded from D_k^- lies at a distance 49ρ from $\mathcal{C}_k^-$, ensuring the contour is strictly isolated.

Recall from (9.12) that

$$A_0(z) := \mathrm{diag}(\alpha_1(z), \ldots, \alpha_r(z)) \quad \text{with} \quad \alpha_l(z) := \lambda_l^{-1} - u_l^{\mathrm{T}} \Phi(z) u_l.$$

The main goal is to establish an analogous result to Lemma 9.3:

LEMMA 9.4. *Under the assumption of Theorem 6, we have*

$$\sup_{z \in \mathcal{C}_j^-} \|A_0(z)^{-1}(K(z) - A_0(z))\| < 1, \qquad \sup_{z \in \mathcal{C}_j^-} \|K(z)^{-1} B(z)\| < 1.$$

Because the boundary $\mathcal{C}_j^-$ maintains a distance of at least ρ_j from any enclosed root ϑ_l, the uniform estimates established in the **Claim** of Section 9.1.1 hold identically for all $z \in \mathcal{C}_j^-$. Consequently, Lemma 9.4 holds.

Hence, by Lemma 9.4 and Lemma 9.1, $\det(A_0(z))$, $g(z) = \det(K(z))$, and $f(z) = \det(K(z) - B(z))$ have the same number of zeros inside the region D_j^-.

Since $\det(A_0(z)) = \prod_{l=1}^{r} \alpha_l(z)$, its zeros in the supercritical regime occur at ϑ_l's. By construction, D_j^- contains exactly j such roots: $\vartheta_1, \ldots, \vartheta_j$. Therefore, $f(z)$ has exactly j zeros inside D_j^-. As noted previously, the zeros of $f(z)$ outside $[-\|E\|, \|E\|]$ correspond exactly to the eigenvalues of the perturbed matrix $\widetilde{A}$. Since the eigenvalues are strictly ordered (i.e., $\widetilde{\lambda}_1 \geqslant \cdots \geqslant \widetilde{\lambda}_r$), these j enclosed roots inside D_j^- must be exactly the top j perturbed eigenvalues $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_j$. Because all j of these eigenvalues lie strictly inside D_j^-, their locations are bounded from below by the leftmost boundary of the region.

Recalling that $\underline{\vartheta}_j := \min_{1 \leqslant l \leqslant j} \vartheta_l$, the real part of any $z \in D_k^-$ satisfies $\mathrm{Re}(z) \geqslant \underline{\vartheta}_k - \rho_k$. Thus, we conclude that for all $s \in [k]$,

$$\widetilde{\lambda}_s \geqslant \underline{\vartheta}_k - \rho_k.$$

The proof is complete.

A Proofs of Lemmas 5.1, Lemma 5.2, Proposition 5.1, and Lemma 5.4

A.1 Proof of Lemma 5.1

Fix z with $|z|^2 \geqslant 6R_{\max}$.

Proof of Part (i): Define a mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ componentwise as

$$(F(y))_i = \frac{1}{z - \sum_j \sigma_{ij}^2 y_j} = \frac{1}{z - (\Sigma y)_i}.$$

Consider the metric $\|y\|_\infty = \max_i |y_i|$ on $\mathbb{C}^n$. Note that ϕ is a fixed point of F, i.e., $F(\phi) = \phi$. Define a closed set

$$\mathcal{B} := \left\{ y \in \mathbb{C}^n : \|y\|_\infty \leqslant \frac{3}{2|z|} \right\}.$$

We first show $F(\mathcal{B}) \subseteq \mathcal{B}$. For $y \in \mathcal{B}$ and every $i \in [n]$,

$$|(\Sigma y)_i| = \Big|\sum_j \sigma_{ij}^2 y_j\Big| \leqslant \|y\|_\infty R_{\max} \leqslant \frac{2}{|z|} R_{\max} \leqslant \frac{|z|}{3}$$

by our supposition $|z|^2 \geqslant 6R_{\max}$. Hence,

$$\big|z - (\Sigma y)_i\big| \geqslant |z| - \big|(\Sigma y)_i\big| \geqslant \frac{2}{3}|z| \tag{A.1}$$

and

$$\big|\,(F(y))_i\,\big| = \frac{1}{|z - (\Sigma y)_i|} \leqslant \frac{3}{2|z|}.$$

This shows that $F(y) \in \mathcal{B}$.

Next, we show that $F : \mathcal{B} \to \mathcal{B}$ is a contraction. Take $y, y' \in \mathcal{B}$. For each $i \in [n]$, using (A.1), we have

$$\begin{aligned}\big|\,(F(y))_i - \big(F(y')\big)_i\,\big| &= \left|\frac{\sum_j \sigma_{ij}^2 (y_j - y_j')}{(z - (\Sigma y)_i)(z - (\Sigma y')_i)}\right| \\ &\leqslant \frac{R_{\max}\|y - y'\|_\infty}{(2|z|/3)^2} = \frac{9}{4}\frac{R_{\max}}{|z|^2}\|y - y'\|_\infty \leqslant \frac{3}{8}\|y - y'\|_\infty.\end{aligned}$$

Since $(\mathcal{B}, \|\cdot\|_\infty)$ is a complete metric space, by Banach's fixed point theorem, F has a unique fixed point in $y^* \in \mathcal{B}$. Since the solution ϕ to the QVE (5.1) is unique and $\phi(z) \to 0$ as $|z| \to \infty$. Therefore, $\phi = y^* \in \mathcal{B}$. In particular,

$$\max_i |\phi_i(z)| \leqslant \frac{3}{2|z|}. \tag{A.2}$$

We have

$$\Big|\big(\Sigma\phi(z)\big)_i\Big| = \Big|\sum_j \sigma_{ij}^2 \phi_j(z)\Big| \leqslant \frac{3}{2|z|} R_{\max}. \tag{A.3}$$

Hence, $\left|\sum_j \sigma_{ij}^2 \phi_j(z)\right| \leqslant \frac{1}{4}|z|$ and

$$\frac{3}{4}|z| \leqslant \big|z - \sum_j \sigma_{ij}^2 \phi_j(z)\big| \leqslant \frac{5}{4}|z|.$$

To show the bounds on $\phi'(z)$, differentiating the equation $\phi_i(z)^{-1} = z - \sum_{j=1}^n \sigma_{ij}^2 \phi_j(z)$ gives

$$-\phi_i(z)^{-2}\phi_i'(z) = 1 - \sum_{j=1}^n \sigma_{ij}^2 \phi_j'(z),$$

hence

$$\phi_i'(z) = -\phi_i(z)^2 \Big(1 - \sum_{j=1}^n \sigma_{ij}^2 \phi_j'(z)\Big)$$

and

$$|\phi_i'(z)| \leqslant |\phi_i(z)|^2 \Big(1 + \sum_{j=1}^n \sigma_{ij}^2 |\phi_j'(z)|\Big).$$

Let $M_1 := \max_i |\phi_i'(z)|$. By (A.2),

$$M_1 \leqslant \frac{9}{4|z|^2}(1 + R_{\max} M_1).$$

Since $R_{\max}/|z|^2 \leqslant 1/6$, we solve $M_1 \leqslant \frac{9}{4|z|^2} + \frac{3}{8}M_1$ to obtain

$$M_1 = \max_i |\phi_i'(z)| \leqslant \frac{18}{5|z|^2}.$$

To bound $\phi''(z)$, similarly, differentiating the equation one more time yields

$$\phi_i''(z) = -2\phi_i(z)\phi_i'(z)\Big(1 - \sum_{j=1}^n \sigma_{ij}^2 \phi_j'(z)\Big) + \phi_i(z)^2 \sum_{j=1}^n \sigma_{ij}^2 \phi_j''(z).$$

Hence,

$$|\phi_i''(z)| \leqslant 2|\phi_i(z)|\,|\phi_i'(z)|\Big(1 + \sum_{j=1}^n \sigma_{ij}^2 |\phi_j'(z)|\Big) + |\phi_i(z)|^2 \sum_{j=1}^n \sigma_{ij}^2 |\phi_j''(z)|.$$

Let $M_2 := \max_i |\phi_i''(z)|$. Using the previous bounds on M_1 and (A.2), we have

$$M_2 \leqslant 2 \cdot \frac{3}{2|z|} \cdot \frac{18}{5|z|^2} \cdot \Big(1 + \frac{3}{5}\Big) + \frac{9R_{\max}}{4|z|^2} M_2.$$

Thus,

$$M_2 = \max_i |\phi_i''(z)| \leqslant \frac{3456}{125|z|^3} \leqslant \frac{28}{|z|^3}.$$

Proof of Part (ii): In the following, we prove (5.2). From $\phi_i(z) = \frac{1}{z-(\Sigma\phi)_i}$ and the exact identity $\frac{1}{z-x} = \frac{1}{z} + \frac{x}{z(z-x)}$, treating $x = (\Sigma\phi)_i$, we get

$$\phi_i(z) = \frac{1}{z} + \frac{(\Sigma\phi)_i}{z(z-(\Sigma\phi)_i)} = \frac{1}{z} + \frac{(\Sigma\phi)_i}{z}\phi_i(z) := \frac{1}{z} + \delta_i(z), \tag{A.4}$$

where

$$|\delta_i(z)| \leqslant \frac{|(\Sigma\phi)_i|}{|z|}|\phi_i(z)| \leqslant \frac{9}{4}\frac{R_{\max}}{|z|^3}$$

by (A.2) and (A.3). Next, from the algebraic identity $\frac{1}{z-x} = \frac{1}{z} + \frac{x}{z^2} + \frac{x^2}{z^2(z-x)}$, we also have

$$\phi_i(z) = \frac{1}{z} + \frac{(\Sigma\phi)_i}{z^2} + \frac{(\Sigma\phi)_i^2}{z^2}\phi_i(z) := \frac{1}{z} + \frac{(\Sigma\phi)_i}{z^2} + \tilde{\delta}_i(z), \tag{A.5}$$

where, by (A.2) and (A.3), we have the estimate

$$|\tilde{\delta}_i(z)| \leqslant \frac{|(\Sigma\phi)_i|^2}{|z|^2}|\phi_i(z)| \leqslant \frac{27}{8}\frac{R_{\max}^2}{|z|^5}.$$

Now we plug (A.4) into the term $\frac{(\Sigma\phi)_i}{z^2}$ in (A.5):

$$\frac{(\Sigma\phi)_i}{z^2} = \frac{1}{z^2}\sum_j \sigma_{ij}^2\left(\frac{1}{z} + \delta_j(z)\right) = \frac{R_i}{z^3} + \frac{\sum_j \sigma_{ij}^2\delta_j(z)}{z^2}.$$

Observe that

$$\left|\frac{\sum_j \sigma_{ij}^2\delta_j(z)}{z^2}\right| \leqslant \frac{R_{\max}}{|z|^2}\max_j|\delta_j(z)| \leqslant \frac{9}{4}\frac{R_{\max}^2}{|z|^5}.$$

Combining the above estimates, we arrive at the final expansion:

$$\phi_i(z) = \frac{1}{z} + \frac{R_i}{z^3} + \left(\tilde{\delta}_i(z) + \frac{\sum_j \sigma_{ij}^2\delta_j(z)}{z^2}\right) := \frac{1}{z} + \frac{R_i}{z^3} + \varepsilon_i(z)$$

with the error term $\varepsilon_i(z)$ satisfying

$$|\varepsilon_i(z)| \leqslant |\tilde{\delta}_i(z)| + \left|\frac{\sum_j \sigma_{ij}^2\delta_j(z)}{z^2}\right| \leqslant \frac{27}{8}\frac{R_{\max}^2}{|z|^5} + \frac{9}{4}\frac{R_{\max}^2}{|z|^5} = \frac{45}{8}\frac{R_{\max}^2}{|z|^5}.$$

This proves (5.2).

To prove (5.4), we start with

$$\phi_i(z) = \left(\frac{1}{z} + \frac{R_i}{z^3} + \varepsilon_i(z)\right)' = -\frac{1}{z^2} - \frac{3R_i}{z^4} + \varepsilon_i'(z).$$

Recall that

$$\varepsilon_i(z) = \tilde{\delta}_i(z) + \frac{1}{z^2}\sum_j \sigma_{ij}^2\delta_j(z),$$

where

$$\delta_i(z) = \frac{(\Sigma\phi)_i}{z}\phi_i(z) \quad \text{and} \quad \tilde{\delta}_i(z) = \frac{(\Sigma\phi)_i^2}{z^2}\phi_i(z).$$

It remains to bound $|\varepsilon_i'(z)|$. Note that

$$\delta_i'(z) = -\frac{(\Sigma\phi)_i}{z^2}\phi_i(z) + \frac{(\Sigma\phi')_i}{z}\phi_i(z) + \frac{(\Sigma\phi)_i}{z}\phi_i'(z),$$
$$\tilde{\delta}_i'(z) = -2\frac{(\Sigma\phi)_i^2}{z^3}\phi_i(z) + 2\frac{(\Sigma\phi)_i(\Sigma\phi')_i}{z^2}\phi_i(z) + \frac{(\Sigma\phi)_i^2}{z^2}\phi_i'(z).$$

Plugging in the previous estimates

$$|\phi_i(z)| \leqslant \frac{3}{2|z|}, \quad |\phi_i'(z)| \leqslant \frac{18}{5|z|^2}, \quad |(\Sigma\phi)_i| \leqslant \frac{3R_{\max}}{2|z|}, \quad |(\Sigma\phi')_i| \leqslant \frac{18R_{\max}}{5|z|^2}$$

yields

$$|\delta_i'(z)| \leqslant 13.05\frac{R_{\max}}{|z|^4} \quad \text{and} \quad |\tilde{\delta}_i'(z)| \leqslant 31.05\frac{R_{\max}^2}{|z|^6}.$$

Together with $|\delta_i(z)| \leqslant \frac{9}{4}\frac{R_{\max}}{|z|^3}$, we obtain that

$$|\varepsilon_i'(z)| \leqslant |\tilde{\delta}_i'(z)| + \left|-\frac{2}{z^3}\sum_j \sigma_{ij}^2\delta_j(z) + \frac{1}{z^2}\sum_j \sigma_{ij}^2\delta_j'(z)\right| \leqslant 48.6\frac{R_{\max}^2}{|z|^6}.$$

We have proved (5.4).

A.2 Proof of Lemma 5.2

Fix $z \in \mathbb{C}$ with $|z| \geqslant \sqrt{6R_{\max}} > 0$. Let $\sqrt{\cdot}$ denote the principal branch of the complex square root on $\mathbb{C}\backslash(-\infty, 0]$. Recall that in (5.1), $\phi_i(z) = \frac{1}{z-\sum_j \sigma_{ij}^2\phi_j(z)}$. For comparison, for each $i \in [k]$, if $R_i > 0$, define an auxiliary function

$$m_i(z) = \frac{z - \sqrt{z^2 - 4R_i}}{2R_i} = \frac{z}{2R_i}\left(1 - \sqrt{1 - \frac{4R_i}{z^2}}\right)$$

and if $R_i = 0$, define $m_i(z) = \frac{1}{z}$. Then $m_i(z)$ satisfies the quadratic equation

$$m_i(z) = \frac{1}{z - R_i m_i(z)}. \tag{A.6}$$

Set

$$d_i(z) = \phi_i(z) - m_i(z) \quad \text{and} \quad d(z) = (d_1(z), \ldots, d_n(z)).$$

Then

$$\Phi(z) = \text{diag}(m_1(z), \ldots, m_n(z)) + d(z).$$

We can rewrite

$$\varepsilon_i(z) = \phi_i(z) - \frac{1}{z} - \frac{R_i}{z^3} = m_i(z) - \frac{1}{z} - \frac{R_i}{z^3} + d_i(z),$$

which yields

$$\operatorname{osc}(\varepsilon_i(z)) \leqslant \operatorname{osc}\left(m_i(z) - \frac{1}{z} - \frac{R_i}{z^3}\right) + 2\|d(z)\|_\infty. \tag{A.7}$$

We now bound each term on the right-hand side separately.

We start with the first term on the right-hand side of (A.7). Since $|\frac{4R_i}{z^2}| \leqslant \frac{2}{3} < 1$, the binomial series for $\sqrt{1-w}$ is absolutely convergent at $w = \frac{4R_i}{z^2}$. Hence,

$$\sqrt{1-w} = 1 - 2\sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k}\left(\frac{w}{4}\right)^{k+1} = 1 - 2\sum_{k=0}^{\infty} C_k \left(\frac{w}{4}\right)^{k+1},$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ denotes the Catalan number. Applying this expansion

$$m_i(z) = \frac{z}{2R_i}\left(1 - \sqrt{1 - \frac{4R_i}{z^2}}\right),$$

we get

$$\begin{aligned} m_i(z) &= \frac{z}{2R_i}\left[1 - \left(1 - 2\sum_{k=0}^{\infty} C_k \left(\frac{4R_i/z^2}{4}\right)^{k+1}\right)\right] = \frac{z}{R_i}\left(\sum_{k=0}^{\infty} C_k \frac{R_i^{k+1}}{z^{2k+2}}\right) \\ &= \frac{1}{z} + \frac{R_i}{z^3} + \sum_{k=2}^{\infty} C_k \frac{R_i^k}{z^{2k+1}} := \frac{1}{z} + \frac{R_i}{z^3} + v_i(z) \end{aligned} \tag{A.8}$$

and thus

$$\Phi(z) = \frac{1}{z} I + \frac{1}{z^3}\mathcal{R} + \operatorname{diag}(v_i(z))_{1 \leqslant i \leqslant n} + d(z). \tag{A.9}$$

It follows that

$$\begin{aligned} \operatorname{osc}\left(m_i(z) - \frac{1}{z} - \frac{R_i}{z^3}\right) &= \max_{i,j} |v_i(z) - v_j(z)| \\ &= \max_{i,j}\left|\sum_{k=2}^{\infty} C_k \frac{R_i^k - R_j^k}{z^{2k+1}}\right| \\ &\leqslant \left(\sum_{k=2}^{\infty} C_k \frac{k R_{\max}^{k-1}}{|z|^{2k+1}}\right)\operatorname{osc}(\mathcal{R}), \end{aligned}$$

where we used

$$|R_i^k - R_j^k| = |(R_i - R_j)(R_i^{k-1} + R_i^{k-2}R_j + \cdots + R_j^{k-1})| \leqslant \operatorname{osc}(\mathcal{R}) \cdot k R_{\max}^{k-1}.$$

Since $R_{\max}/|z|^2 \leqslant 1/6$ by our assumption, we continue to have

$$\begin{aligned}\operatorname{osc}\left(m_i(z) - \frac{1}{z} - \frac{R_i}{z^3}\right) &\leqslant \frac{R_{\max}}{|z|^5}\left(\sum_{k=2}^{\infty} kC_k \frac{R_{\max}^{k-2}}{|z|^{2(k-2)}}\right)\operatorname{osc}(\mathcal{R}) \\ &\leqslant \frac{R_{\max}}{|z|^5}\operatorname{osc}(\mathcal{R})\left(\sum_{k=2}^{\infty} \frac{kC_k}{6^{k-2}}\right) \\ &\leqslant \frac{11R_{\max}}{|z|^5}\operatorname{osc}(\mathcal{R}). \end{aligned} \tag{A.10}$$

In the last step, we used $\sum_{k=2}^{\infty} \frac{kC_k}{6^{k-2}} \leqslant 11$. This can be derived by taking the generating function of the Catalan numbers

$$c(x) = \sum_{k=0}^{\infty} C_k x^k = \frac{1-\sqrt{1-4x}}{2x}. \tag{A.11}$$

The sum is $S = \sum_{k=2}^{\infty} kC_k x^{k-2}$ evaluated at $x = 1/6$. Differentiating the generating function gives $c'(x) = \sum_{k=1}^{\infty} kC_k x^{k-1} = 1 + xS$. Therefore, the sum is exactly given by $S = \frac{c'(x)-1}{x}$. Taking the derivative yields $c'(x) = \frac{1-2x-\sqrt{1-4x}}{2x^2\sqrt{1-4x}}$. Evaluating this at $x = 1/6$ gives $c'(1/6) = 12\sqrt{3} - 18$. Substituting this back into the sum formula yields the exact value:

$$\sum_{k=2}^{\infty} \frac{kC_k}{6^{k-2}} = 6(12\sqrt{3} - 19) \leqslant 11.$$

Furthermore, we obtain the following bound for later estimates:

$$\begin{aligned}|m_i(z) - m_j(z)| &\leqslant \frac{|R_i - R_j|}{|z|^3} + \operatorname{osc}\left(m_i(z) - \frac{1}{z} - \frac{R_i}{z^3}\right) \\ &\leqslant \frac{\operatorname{osc}(\mathcal{R})}{|z|^3} + \frac{11R_{\max}}{|z|^5}\operatorname{osc}(\mathcal{R}) \\ &\leqslant \left(1 + \frac{1}{6}\right)\frac{\operatorname{osc}(\mathcal{R})}{|z|^3} = \frac{7}{6}\frac{\operatorname{osc}(\mathcal{R})}{|z|^3}. \end{aligned} \tag{A.12}$$

In the last inequality, we applied $R_{\max}/|z|^2 \leqslant 1/6$.

Next, we turn to the second term on the right-hand side of (A.7). For each i, we have

$$\begin{aligned}|d_i(z)| = |\phi_i(z) - m_i(z)| &= \left|\frac{1}{z - \sum_j \sigma_{ij}^2 \phi_j(z)} - \frac{1}{z - R_i m_i(z)}\right| \\ &= \frac{|R_i m_i(z) - \sum_j \sigma_{ij}^2 \phi_j(z)|}{|z - \sum_j \sigma_{ij}^2 \phi_j(z)| \cdot |z - R_i m_i(z)|} \\ &\leqslant \frac{2}{|z|^2}\Big|R_i m_i(z) - \sum_j \sigma_{ij}^2 \phi_j(z)\Big|. \end{aligned} \tag{A.13}$$

For the last step above, observe that $|z - \sum_j \sigma_{ij}^2 \phi_j(z)| \geqslant \frac{3}{4}|z|$ by Lemma 5.1 (i). In addition, we claim that

$$|z - R_i m_i(z)| \geqslant \frac{2|z|}{3}.$$

To see this, from (A.6), we have $|z - R_i m_i(z)| = \frac{1}{|m_i(z)|}$. Using the series expansion of $m_i(z)$ in (A.8) and $R_{\max}/|z|^2 \leqslant 1/6$, we get

$$|m_i(z)| \leqslant \frac{1}{|z|} \sum_{k=0}^{\infty} C_k \Big(\frac{R_i}{|z|^2}\Big)^k \leqslant \frac{1}{|z|} \sum_{k=0}^{\infty} \frac{C_k}{6^k} = \frac{c(1/6)}{|z|} < \frac{1.5}{|z|},$$

where $c(x)$ is defined in (A.11). The claim is proved.

It remains to estimate $|R_i m_i(z) - \sum_j \sigma_{ij}^2 \phi_j(z)|$. We write

$$\begin{aligned}\Big|R_i m_i(z) - \sum_j \sigma_{ij}^2 \phi_j(z)\Big| &= |\sum_j \sigma_{ij}^2 (m_i(z) - m_j(z)) + \sum_j \sigma_{ij}^2 (m_j(z) - \phi_j(z))| \\ &\leqslant \sum_j \sigma_{ij}^2 |m_i(z) - m_j(z)| + R_{\max} \|d(z)\|_\infty.\end{aligned}$$

Plugging in (A.12), we further have

$$|R_i m_i(z) - \sum_j \sigma_{ij}^2 \phi_j(z)| \leqslant \frac{7}{6} \frac{R_{\max}}{|z|^3} \mathrm{osc}(\mathcal{R}) + R_{\max} \|d(z)\|_\infty$$

and thus from (A.13),

$$\begin{aligned}\|d(z)\|_\infty = \max_i |d_i(z)| &\leqslant \frac{2}{|z|^2} \left(\frac{7}{6} \frac{R_{\max}}{|z|^3} \mathrm{osc}(\mathcal{R}) + R_{\max} \|d(z)\|_\infty\right) \\ &\leqslant \frac{7}{3} \frac{R_{\max}}{|z|^5} \mathrm{osc}(\mathcal{R}) + \frac{1}{3} \|d(z)\|_\infty.\end{aligned}$$

Rearranging terms, we arrive at

$$\|d(z)\|_\infty \leqslant \frac{7 R_{\max}}{2|z|^5} \mathrm{osc}(\mathcal{R}). \tag{A.14}$$

Finally, we conclude that

$$\mathrm{osc}(\varepsilon_i(z)) \leqslant \mathrm{osc}\left(m_i(z) - \frac{1}{z} - \frac{R_i}{z^3}\right) + 2\|d(z)\|_\infty \leqslant 14.5 \frac{R_{\max}}{z^5} \mathrm{osc}(\mathcal{R}).$$

A.3 Proof of Proposition 5.1

By part (ii) of Lemma 5.1,

$$u_j^{\mathrm{T}} \Phi(z) u_j = \frac{1}{z} + \frac{\mathcal{V}_{jj}}{z^3} + \epsilon_j(z), \qquad |\epsilon_j(z)| = |u_j^{\mathrm{T}} \varepsilon(z) u_j| \leqslant \frac{45}{8} \frac{R_{\max}^2}{z^5},$$

for all real $z \geqslant 3\sqrt{R_{\max}}$. For brevity, denote $f(z) = u_j^{\mathrm{T}} \Phi(z) u_j$ and thus $\widehat{\alpha}_j(z) = 1 - \lambda_j f(z)$.

We first show that $\widehat{\alpha}_j$ has a unique zero. Note that $\mathcal{V}_{jj} \geqslant 0$. At $z_0 := 3\sqrt{R_{\max}}$,

$$f(z_0) = u_j^{\mathrm{T}}\Phi(z_0)u_j \geqslant \frac{1}{z_0} - \frac{45}{8}\frac{R_{\max}^2}{z_0^5} = \left(\frac{1}{3} - \frac{45}{8\cdot 3^5}\right)\frac{1}{\sqrt{R_{\max}}} = \frac{67}{216}\frac{1}{\sqrt{R_{\max}}},$$

hence

$$\widehat{\alpha}_j(z_0) \leqslant 1 - \frac{67}{216}\frac{\lambda_j}{\sqrt{R_{\max}}} < 0$$

since $\lambda_j \geqslant 6\sqrt{R_{\max}}$. On the other hand, $\widehat{\alpha}_j(z) \to 1$ as $z \to \infty$. Next, by part (ii) of Lemma 5.1,

$$f'(z) = u_j^{\mathrm{T}}\Phi'(z)u_j = -\frac{1}{z^2} - \frac{3\mathcal{V}_{jj}}{z^4} + \epsilon_j'(z), \qquad |\epsilon_j'(z)| = |u_j^{\mathrm{T}}\varepsilon'(z)u_j| \leqslant \frac{243}{5}\frac{R_{\max}^2}{z^6} \tag{A.15}$$

Therefore, on the interval $[3\sqrt{R_{\max}}, \infty)$,

$$\widehat{\alpha}_j'(z) = -f'(z) \geqslant \lambda_j\left(\frac{1}{z^2} - \frac{243}{5}\frac{R_{\max}^2}{z^6}\right) > 0$$

since

$$\frac{243}{5}\frac{R_{\max}^2}{z^6} \leqslant \frac{243}{5}\frac{1}{81z^2} = \frac{243}{405}\frac{1}{z^2} < \frac{1}{z^2}.$$

Therefore, $\widehat{\alpha}_j$ is strictly increasing, and has a unique and simple real zero $\vartheta_j \in [3\sqrt{R_{\max}}, \infty)$.

Observe that at $z = \lambda_j/2 \geqslant 3\sqrt{R_{\max}}$ and $z = 2\lambda_j$, we have

$$\begin{aligned}\widehat{\alpha}_j(\lambda_j/2) = 1 - \lambda_j u_j^{\mathrm{T}}\Phi(\lambda_j/2)u_j &\leqslant 1 - \lambda_j\left(\frac{2}{\lambda_j} - \frac{45}{8}\frac{2^5R_{\max}^2}{\lambda_j^5}\right)\\ &\leqslant -1 + \frac{5}{36} < 0\end{aligned}$$

and

$$\begin{aligned}\widehat{\alpha}_j(2\lambda_j) = 1 - \lambda_j u_j^{\mathrm{T}}\Phi(2\lambda_j)u_j &\geqslant 1 - \lambda_j\left(\frac{1}{2\lambda_j} + \frac{45}{8}\frac{R_{\max}^2}{2^5\lambda_j^5}\right)\\ &\geqslant \frac{1}{2} - \frac{45}{9216} > 0.\end{aligned}$$

Since $\widehat{\alpha}_j$ is strictly increasing, we have

$$\frac{\lambda_j}{2} \leqslant \vartheta_j \leqslant 2\lambda_j.$$

Now we locate ϑ_j. Note that $f(\vartheta_j) = \lambda_j^{-1}$. We introduce a scalar comparison function

$$m_{\mathcal{V},j}(z) = \frac{z - \sqrt{z^2 - 4\mathcal{V}_{jj}}}{2\mathcal{V}_{jj}}$$

which satisfies the quadratic equation

$$m(z) = \frac{1}{z - \mathcal{V}_{jj}m(z)}. \tag{A.16}$$

This function approximates $f(z) = u_j^{\mathrm{T}}\Phi(z)u_j$ by noting $\mathcal{V}_{jj} = \sum_i u_j(i)^2 R_i$ and the definition of $\phi_i(z)$ in (5.1).

Denote

$$x_j := \lambda_j + \frac{\mathcal{V}_{jj}}{\lambda_j}.$$

Then x_j is a zero of $1 - \lambda_j m_{\mathcal{V},j}(z) = 0$. Indeed, plugging $m_{\mathcal{V},j}(z) = \frac{1}{\lambda_j}$ into (A.16), we get $\lambda_j = z - \mathcal{V}_{jj} m_{\mathcal{V},j}(z) = z - \frac{\mathcal{V}_{jj}}{\lambda_j}$ and thus $z = \lambda_j + \frac{\mathcal{V}_{jj}}{\lambda_j}$.

Next, we expand $m_{\mathcal{V},j}(z)$ as a Catalan series the same as (A.8) at $z = x_j$:

$$\frac{1}{\lambda_j} = m_{\mathcal{V},j}(x_j) = \frac{1}{x_j} + \frac{\mathcal{V}_{jj}}{x_j^3} + \sum_{k=2}^{\infty} C_k \frac{\mathcal{V}_{jj}^k}{x_j^{2k+1}}. \tag{A.17}$$

To estimate $|\vartheta_j - x_j|$, we first show that $f(x_j) = u_j^{\mathrm{T}}\Phi(x_j)u_j \approx \frac{1}{\lambda_j} = f(\vartheta_j)$. Then we apply the mean value theorem to bound $|\vartheta_j - x_j|$.

In the proof of Lemma 5.2, we have the expansion of $\Phi(z)$ given in (A.9). Hence,

$$f(x_j) = u_j^{\mathrm{T}}\Phi(x_j)u_j = \frac{1}{x_j} + \frac{\mathcal{V}_{jj}}{x_j^3} + \sum_{k=2}^{\infty} C_k \frac{\sum_i u_j(i)^2 R_i^k}{x_j^{2k+1}} + u_j^{\mathrm{T}} d(x_j) u_j.$$

Subtracting with (A.17), we obtain

$$|f(x_j) - f(\vartheta_j)| = |f(x_j) - \frac{1}{\lambda_j}| \leqslant \sum_{k=2}^{\infty} \frac{C_k}{x_j^{2k+1}} \left| \mathcal{V}_{jj}^k - \sum_i u_j(i)^2 R_i^k \right| + \|d(x_j)\|_\infty.$$

Note that

$$\begin{aligned}\left| \mathcal{V}_{jj}^k - \sum_i u_j(i)^2 R_i^k \right| \leqslant \sum_i u_j(i)^2 \left| \mathcal{V}_{jj}^k - R_i^k \right| &\leqslant k R_{\max}^k \sum_i u_j(i)^2 \left| \mathcal{V}_{jj} - R_i \right| \\ &\leqslant k R_{\max}^k \operatorname{osc}(\mathcal{R}).\end{aligned}$$

Following the same step as (A.10) and combining the bound on $\|d\|_\infty$ in (A.14), we arrive at

$$|f(x_j) - f(\vartheta_j)| \leqslant 11 \frac{R_{\max}}{x_j^5} \operatorname{osc}(\mathcal{R}) + \frac{7}{2} \frac{R_{\max}}{x_j^5} \operatorname{osc}(\mathcal{R}) \leqslant 14.5 \frac{R_{\max}}{\lambda_j^5} \operatorname{osc}(\mathcal{R}).$$

Finally, by mean value theorem, $|f(x_j) - f(\vartheta_j)| = |x_j - \vartheta_j| \cdot |f'(w)|$ for some w between x_j and ϑ_j. Since $\lambda_j/2 \leqslant \vartheta_j \leqslant 2\lambda_j$ and $x_j \leqslant 2\lambda_j$, we have $w \leqslant 2\lambda_j$. By (A.15), we get

$$f'(w) \geqslant \frac{1}{w^2} - \frac{243}{5} \frac{R_{\max}^2}{w^6} \geqslant \frac{1}{w^2} \left(1 - \frac{243}{5 \cdot 81} \right) = \frac{2}{5w^2} \geqslant \frac{1}{10\lambda_j^2}.$$

Rearranging yields that

$$\left| \lambda_j + \frac{\mathcal{V}_{jj}}{\lambda_j} - \vartheta_j \right| = |x_j - \vartheta_j| \leqslant 145 \frac{R_{\max}}{\lambda_j^3} \operatorname{osc}(\mathcal{R}).$$

The proof is now complete.

A.4 Proof of Lemma 5.4

We apply Corollary 3.12 from [14] with a standard truncation technique. Denote

$$L_n := K\sigma_{\max}((D+4)\log n)^{1/2}.$$

By a union bound, we have

$$\mathbb{P}(\max_{i,j}|E_{ij}| > L_n) \leqslant n^2 \cdot 2\exp(-(L_n/K)^2) = 2n^{-(D+2)}.$$

We split E into big and small parts according to L_n. Namely, we write $E = Z + M + B$, where we define these symmetric matrices via

$$Z_{ij} = E_{ij}\mathbf{1}_{\{|E_{ij}|\leqslant L_n\}} - \mathbb{E}(E_{ij}\mathbf{1}_{\{|E_{ij}|\leqslant L_n\}}),\ M_{ij} = \mathbb{E}(E_{ij}\mathbf{1}_{\{|E_{ij}|\leqslant L_n\}}),\ B_{ij} = E_{ij}\mathbf{1}_{\{|E_{ij}|>L_n\}}.$$

Note that if $\max_{i,j}|E_{ij}| \leqslant L_n$, then $B = 0$. We work on the event $\mathcal{E} = \{\max_{i,j}|E_{ij}| \leqslant L_n\}$ below and previous calculations suggest that $\mathbb{P}(\mathcal{E}) \geqslant 1 - 2n^{-(D+2)}$.

Next, we bound $\|M\|$. Note that since E_{ij} has mean 0, $M_{ij} = -\mathbb{E}(E_{ij}\mathbf{1}_{\{|E_{ij}|>L_n\}})$. Therefore,

$$|M_{ij}| \leqslant \mathbb{E}(|E_{ij}|\mathbf{1}_{\{|E_{ij}|>L_n\}}) = \int_{L_n}^{\infty} \mathbb{P}(|E_{ij}| \geqslant t)\,dt \leqslant 2L_n n^{-(D+4)}$$

since $\mathbb{P}(|E_{ij}| > L_n) \leqslant 2n^{-(D+4)}$. Then we have

$$\|M\| \leqslant \max_i \sum_j |M_{ij}| \leqslant 2L_n n^{-(D+3)}.$$

Finally, for the bound on $\|Z\|$, we apply Corollary 3.12 from [14]. Note that Z_{ij}'s have mean 0 and $|Z_{ij}| \leqslant 2L_n$. Also, $\mathbb{E}(Z_{ij}^2) \leqslant \mathbb{E}(E_{ij}^2) = \sigma_{ij}^2$. Applying of [14, Corollary 3.12], together with a standard symmetrization (as in the proof of [14, Corollary 3.3]), yields that for $t > 0$,

$$\mathbb{P}\left(\|Z\| \geqslant 2\sqrt{2}(1+\epsilon)\tilde{\sigma} + t\right) \leqslant n\exp(-t^2/(c_\epsilon L_n)^2).$$

We choose ϵ such that $2\sqrt{2}(1+\epsilon) = 2.9$ and denote $c_\epsilon = c$. The parameter $\tilde{\sigma} = \max_i \sqrt{\sum_j \mathbb{E}(Z_{ij}^2)} = \sqrt{\max_i \sigma_{ij}^2} = \sqrt{R_{\max}}$ by our supposition. Choosing $t = cL_n\sqrt{(D+3)\log n}$, we obtain that

$$\mathbb{P}\left(\|Z\| \geqslant 2.9\sqrt{R_{\max}} + cL_n\sqrt{(D+3)\log n}\right) \leqslant n^{-(D+2)}.$$

Combining the above estimates, we have that with probability at least $1 - 3n^{-(D+2)}$,

$$\begin{aligned} \|E\| \leqslant \|Z\| + \|M\| + \|B\| &\leqslant 2L_n n^{-(D+3)} + 1.5\sqrt{R_{\max}} + cL_n\sqrt{(D+3)\log n} \\ &\leqslant 2.9\sqrt{R_{\max}} + cK\sigma_{\max}(D+4)\log n. \end{aligned}$$

B Proofs of Proposition 7.1 and Proposition 7.2

B.1 Proof of Proposition 7.1

Let $Q = I - UU^{\mathrm{T}}$ be the orthogonal projection onto the null space of A. Set $P_J = U_J U_J^{\mathrm{T}}$ where $J = [r]\backslash[k]$ and $U_J = (u_{k+1}, \ldots, u_r)$. Then

$$P_{U_k^{\perp}} = P_J + Q.$$

We first prove (7.1) in part (i). Starting from $\|\sin\angle(U_k, \widetilde{U}_k)\| = \|P_{U_k^{\perp}} P_{\widetilde{U}_k}\|$ (see, for instance, [25, Exercises VII. 1. 9-1.11]), we have

$$\begin{aligned}\|\sin\angle(U_k, \widetilde{U}_k)\| = \|(P_J + Q)P_{\widetilde{U}_k}\| &\leqslant \|P_J P_{\widetilde{U}_k}\| + \|QP_{\widetilde{U}_k}\| \\ &\leqslant \|U_J^{\mathrm{T}}\widetilde{U}_k\|_F + \|QP_{\widetilde{U}_k}\|.\end{aligned}$$

It remains to bound $\|QP_{\widetilde{U}_k}\| = \|Q\widetilde{U}_k\|$. From the spectral decomposition of $\widetilde{A}$, we have $(A + E)\widetilde{U}_k = \widetilde{U}_k\tilde{\Lambda}_k$. Multiplying by Q on the left and noting that $QA = 0$, we obtain $QE\widetilde{U}_k = Q\widetilde{U}_k\widetilde{\Lambda}_k$. Since $\widetilde{\Lambda}_k$ is invertible, we have

$$Q\widetilde{U}_k = QE\widetilde{U}_k\widetilde{\Lambda}_k^{-1}.$$

To see that $\widetilde{\Lambda}_k$ is indeed invertible, note that by Weyl's inequality, for every $1 \leqslant s \leqslant k$,

$$\frac{3}{2}\lambda_s \geqslant \lambda_s + \|E\| \geqslant \widetilde{\lambda}_s \geqslant \lambda_s - \|E\| \geqslant \frac{1}{2}\lambda_s, \tag{B.1}$$

where the last inequality follows from our assumption $\lambda_s \geqslant \lambda_k \geqslant 2\|E\|$. Therefore,

$$\|QP_{\widetilde{U}_k}\| = \|QE\widetilde{U}_k\widetilde{\Lambda}_k^{-1}\| \leqslant \|QE\widetilde{U}_k\| \cdot \|\widetilde{\Lambda}_k^{-1}\| \leqslant \frac{\|E\|}{\widetilde{\lambda}_k} \leqslant 2\frac{\|E\|}{\lambda_k}. \tag{B.2}$$

Now we turn to part (ii). Note that (7.2) is given in Eq. (10) of [83]. It remains to prove (7.3). It follows from the decomposition $\widetilde{u}_s = P_{U_k}\widetilde{u}_s + P_J\widetilde{u}_s + Q\widetilde{u}_s$ for each $1 \leqslant s \leqslant k$ that

$$\widetilde{U}_k = P_{U_k}\widetilde{U}_k + P_J\widetilde{U}_k + Q\widetilde{U}_k$$

and thus

$$\begin{aligned}\|\widetilde{U}_k - P_{U_k}\widetilde{U}_k\|_{2,\infty} = \|P_J\widetilde{U}_k + Q\widetilde{U}_k\|_{2,\infty} &\leqslant \|U_J U_J^{\mathrm{T}}\widetilde{U}_k\|_{2,\infty} + \|Q\widetilde{U}_k\|_{2,\infty} \\ &\leqslant \|U\|_{2,\infty} \cdot \|U_J^{\mathrm{T}}\widetilde{U}_k\|_F + \|Q\widetilde{U}_k\|_{2,\infty}.\end{aligned}$$

To finish the proof, we show that

$$\|Q\widetilde{U}_k\|_{2,\infty} \leqslant \frac{13}{2}\sqrt{k}\|U\|_{2,\infty}\frac{R_{\max}}{\widetilde{\lambda}_k^2} + 4\sqrt{\sum_{s=1}^{k}\widetilde{\lambda}_s^2 \max_{1\leqslant i\leqslant n}\left\|e_i^{\mathrm{T}}Q\Xi(z)U\right\|^2}.$$

For $1 \leqslant s \leqslant k$, from the $(A+E)\widetilde{u}_s = \widetilde{\lambda}_s \widetilde{u}_s$, we have $A\widetilde{u}_s = (\widetilde{\lambda}_s - E)\widetilde{u}_s$. Therefore,

$$\widetilde{u}_s = G(\widetilde{\lambda}_s) A\widetilde{u}_s = \Phi(\widetilde{\lambda}_s) A\widetilde{u}_s + \Xi(\widetilde{\lambda}_s) A\widetilde{u}_s \tag{B.3}$$

and for any canonical vector e_l, we obtain

$$e_l^{\mathrm{T}} Q \widetilde{U}_k = \left(e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) A\widetilde{u}_s\right)_{s\in[k]} + \left(e_l^{\mathrm{T}} Q \Xi(\widetilde{\lambda}_s) A\widetilde{u}_s\right)_{s\in[k]} := \mathsf{q}_l^{(1)} + \mathsf{q}_l^{(2)}.$$

It follows that

$$\|Q\widetilde{U}_k\|_{2,\infty} = \max_{1\leqslant l\leqslant n} \|e_l^{\mathrm{T}} Q\widetilde{U}_k\| \leqslant \max_{1\leqslant l\leqslant n} \left(\|\mathsf{q}_l^{(1)}\| + \|\mathsf{q}_l^{(2)}\|\right).$$

We bound $\|\mathsf{q}_l^{(1)}\|$ and $\|\mathsf{q}_l^{(2)}\|$ on the right hand side. We start from

$$\|\mathsf{q}_l^{(1)}\| = \sqrt{\sum_{s=1}^{k} \left(e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) A\widetilde{u}_s\right)^2}.$$

For each $l \in [n]$, observe that

$$|e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) A\widetilde{u}_s| = |e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) U \cdot \Lambda U^{\mathrm{T}} \widetilde{u}_s| \leqslant \|e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) U\| \cdot \|\Lambda U^{\mathrm{T}} \widetilde{u}_s\|. \tag{B.4}$$

Note that

$$\|\Lambda U^{\mathrm{T}} \widetilde{u}_s\| \leqslant 2\widetilde{\lambda}_s.$$

This is because from $(A+E)\widetilde{u}_s = \widetilde{\lambda}_s \widetilde{u}_s$, we have $A\widetilde{u}_s = U\Lambda U^{\mathrm{T}}\widetilde{u}_s = (\widetilde{\lambda}_s I - E)\widetilde{u}_s$. Multiplying U^{T} on each side, we further get $\Lambda U^{\mathrm{T}}\widetilde{u}_s = U^{\mathrm{T}}(\widetilde{\lambda}_s - E)\widetilde{u}_s$ and

$$\|\Lambda U^{\mathrm{T}} \widetilde{u}_s\| \leqslant (\widetilde{\lambda}_s + \|E\|) \leqslant 2\widetilde{\lambda}_s$$

by (B.1) and our assumption $\lambda_s \geqslant \lambda_k \geqslant 2\|E\|$.

Next, we estimate $\|e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) U\|$. Since $\widetilde{\lambda}_s \geqslant \frac{1}{2}\lambda_s \geqslant 3\sqrt{R_{\max}}$, we plug in the expansion of $\Phi(\widetilde{\lambda}_s)$ in (5.3) to get

$$e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) U = e_l^{\mathrm{T}} Q \left(\frac{1}{\widetilde{\lambda}_s} I_n + \frac{1}{\widetilde{\lambda}_s^3}\mathcal{R} + \varepsilon(\widetilde{\lambda}_s)\right) U = e_l^{\mathrm{T}} Q \left(\frac{1}{\widetilde{\lambda}_s^3}\mathcal{R} + \varepsilon(\widetilde{\lambda}_s)\right) U.$$

Note that both $\mathcal{R}$ and $\varepsilon(\widetilde{\lambda}_s)$ are diagonal matrices. Indeed, for a general diagonal matrix $D = \operatorname{diag}(d_1, \cdots, d_n)$,

$$e_l^{\mathrm{T}} Q D U = e_l^{\mathrm{T}} (I - UU^{\mathrm{T}}) D U = e_l^{\mathrm{T}} D U - e_l^{\mathrm{T}} U (U^{\mathrm{T}} D U) = e_l^{\mathrm{T}} U (d_l I - U^{\mathrm{T}} D U)$$

due to $e_l^{\mathrm{T}} D = d_l e_l^{\mathrm{T}}$ and hence,

$$\|e_l^{\mathrm{T}} Q D U\| \leqslant \|e_l^{\mathrm{T}} U\| (|d_l| + \|U^{\mathrm{T}} D U\|) \leqslant 2\|U\|_{2,\infty} \|D\|.$$

Using the error estimate in (5.3), we have

$$\|e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) U\| \leqslant 2\|U\|_{2,\infty} \left(\frac{R_{\max}}{\widetilde{\lambda}_s^3} + \frac{45}{8}\frac{R_{\max}^2}{\widetilde{\lambda}_s^5}\right) \leqslant \frac{13}{4}\|U\|_{2,\infty} \frac{R_{\max}}{\widetilde{\lambda}_s^3},$$

where we used $\widetilde{\lambda}_s \geqslant \frac{1}{2}\lambda_s \geqslant 3\sqrt{R_{\max}}$ from (B.1) and our assumption $\lambda_k \geqslant 6\sqrt{R_{\max}}$. Combining the above estimates, we continue from (B.4) to achieve

$$|e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) A \widetilde{u}_s| \leqslant \frac{13}{2}\|U\|_{2,\infty}\frac{R_{\max}}{\widetilde{\lambda}_s^2} \leqslant \frac{13}{2}\|U\|_{2,\infty}\frac{R_{\max}}{\widetilde{\lambda}_k^2}.$$

Thus

$$\|\mathsf{q}_l^{(1)}\| = \sqrt{\sum_{s=1}^{k}\left(e_l^{\mathrm{T}} Q \Phi(\widetilde{\lambda}_s) A \widetilde{u}_s\right)^2} \leqslant \frac{13}{2}\sqrt{k}\|U\|_{2,\infty}\frac{R_{\max}}{\widetilde{\lambda}_k^2}.$$

The estimate for

$$\|\mathsf{q}_l^{(2)}\| = \sqrt{\sum_{s=1}^{k}\left(e_l^{\mathrm{T}} Q \Xi(\widetilde{\lambda}_s) A \widetilde{u}_s\right)^2}$$

is simpler. Using an estimate similar to (B.4), we have

$$\begin{aligned}|e_l^{\mathrm{T}} Q \Xi(\widetilde{\lambda}_s) A \widetilde{u}_s| = |e_l^{\mathrm{T}} Q \Xi(\widetilde{\lambda}_s) U \cdot \Lambda U^{\mathrm{T}} \widetilde{u}_s| &\leqslant \|e_l^{\mathrm{T}} Q \Xi(\widetilde{\lambda}_s) U\| \cdot \|\Lambda U^{\mathrm{T}} \widetilde{u}_s\| \\ &\leqslant 2\widetilde{\lambda}_s \|e_l^{\mathrm{T}} Q \Xi(\widetilde{\lambda}_s) U\|.\end{aligned}$$

Thus

$$\|\mathsf{q}_l^{(2)}\| \leqslant 4\sqrt{\sum_{s=1}^{k}\widetilde{\lambda}_s^2 \max_{1\leqslant i\leqslant n}\left\|e_i^{\mathrm{T}} Q \Xi(z) U\right\|^2}.$$

Combining all the above estimates, we complete the proof.

B.2 Proof of Proposition 7.2

Fix $s \in [k]$. Recall that $J = [r]\backslash[k]$. We partition J into two parts

$$\mathcal{J} := \{k+1, \ldots, r_+\} \quad \text{and} \quad \mathcal{N} := \{r_+ + 1, \ldots, r\}.$$

Then

$$\|U_J^{\mathrm{T}} \widetilde{u}_s\|^2 = \|U_{\mathcal{J}}^{\mathrm{T}} \widetilde{u}_s\|^2 + \|U_{\mathcal{N}}^{\mathrm{T}} \widetilde{u}_s\|^2.$$

We estimate each term separately.

Step 1. Bound on $\|U_{\mathcal{J}}^{\mathrm{T}} \widetilde{u}_s\|$. Set $\mathcal{I} = [k] \cup \{r_+ + 1, \ldots, r\} = [r]\backslash\mathcal{J}$. We then have the decomposition

$$A = U\Lambda U^{\mathrm{T}} = U_{\mathcal{J}}\Lambda_{\mathcal{J}}U_{\mathcal{J}}^{\mathrm{T}} + U_{\mathcal{I}}\Lambda_{\mathcal{I}}U_{\mathcal{I}}^{\mathrm{T}}.$$

Multiplying both sides of (B.3) by $U_{\mathcal{J}}^{\mathrm{T}}$ from the left yields

$$\begin{aligned}U_{\mathcal{J}}^{\mathrm{T}} \widetilde{u}_s &= U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) A \widetilde{u}_s + U_{\mathcal{J}}^{\mathrm{T}} \Xi(\widetilde{\lambda}_s) A \widetilde{u}_s \\ &= U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{J}} \Lambda_{\mathcal{J}} \cdot U_{\mathcal{J}}^{\mathrm{T}} \widetilde{u}_s + U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{I}} \cdot \Lambda_{\mathcal{I}} U_{\mathcal{I}}^{\mathrm{T}} \widetilde{u}_s + U_{\mathcal{J}}^{\mathrm{T}} \Xi(\widetilde{\lambda}_s) U \cdot \Lambda U^{\mathrm{T}} \widetilde{u}_s,\end{aligned}$$

where we have used the decomposition of A. Isolating the term $U_{\mathcal{J}}^{\mathrm{T}} \widetilde{u}_s$ gives

$$\left(I - U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{J}} \Lambda_{\mathcal{J}}\right) U_{\mathcal{J}}^{\mathrm{T}} \widetilde{u}_s = U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{I}} \cdot \Lambda_{\mathcal{I}} U_{\mathcal{I}}^{\mathrm{T}} \widetilde{u}_s + U_{\mathcal{J}}^{\mathrm{T}} \Xi(\widetilde{\lambda}_s) U \cdot \Lambda U^{\mathrm{T}} \widetilde{u}_s. \quad \text{(B.5)}$$

Denote

$$L_{\mathcal{J}}(\widetilde{\lambda}_s) := I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}}.$$

We first bound the smallest singular value $s_{\min}(L_{\mathcal{J}}(\widetilde{\lambda}_s))$. We split

$$\begin{aligned} U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}} &= \mathrm{diag}(u_j^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)u_j)_{j\in\mathcal{J}} + \mathrm{Off}(U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}) \\ &:= \mathrm{diag}(u_j^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)u_j)_{j\in\mathcal{J}} + Q_{\mathcal{J}}(\widetilde{\lambda}_s) \end{aligned}$$

and rewrite

$$\begin{aligned} L_{\mathcal{J}}(\widetilde{\lambda}_s) &= I - \mathrm{diag}(u_j^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)u_j)_{j\in\mathcal{J}}\Lambda_{\mathcal{J}} - Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}} \\ &= \mathrm{diag}\left(1 - \lambda_j u_j^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)u_j\right)_{j\in\mathcal{J}} - Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}. \end{aligned}$$

Then

$$s_{\min}(L_{\mathcal{J}}(\widetilde{\lambda}_s)) \geqslant \frac{\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s)}{\widetilde{\lambda}_s} - \|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}\| \tag{B.6}$$

with $\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s) = \widetilde{\lambda}_s \min_{j\in\mathcal{J}} |1 - \lambda_j u_j^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)u_j|$ defined in (7.4).

Now we bound $\|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}\|$ on the right-hand side of (B.6). Since $\widetilde{\lambda}_s \geqslant 3\sqrt{R_{\max}}$, by the second-order expansion of Φ in (A.5), we have

$$Q_{\mathcal{J}}(\widetilde{\lambda}_s) = \mathrm{Off}(U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}) = \mathrm{Off}\left(\frac{\mathcal{V}_{\mathcal{J}\mathcal{J}}}{\widetilde{\lambda}_s^3}\right) + \mathrm{Off}\left(U_{\mathcal{J}}^{\mathrm{T}}\varepsilon(\widetilde{\lambda}_s)U_{\mathcal{J}}\right).$$

Note, by orthogonality, that

$$\left\|\mathrm{Off}\left(U_{\mathcal{J}}^{\mathrm{T}}\varepsilon(\widetilde{\lambda}_s)U_{\mathcal{J}}\right)\right\| = \left\|\mathrm{Off}\left(U_{\mathcal{J}}^{\mathrm{T}}\big(\varepsilon(\widetilde{\lambda}_s) - c^*I\big)U_{\mathcal{J}}\right)\right\| \leqslant \|\varepsilon(\widetilde{\lambda}_s) - c^*I\| = \frac{1}{2}\mathrm{osc}(\varepsilon(\widetilde{\lambda}_s)),$$

where we select $c^* = \frac{1}{2}\left(\max_i \varepsilon_i(\widetilde{\lambda}_s) + \min_i \varepsilon_i(\widetilde{\lambda}_s)\right)$. It follows, by applying Proposition 5.2, that

$$\|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{L}}\| \leqslant \frac{\lambda_{k+1}}{\widetilde{\lambda}_s^3}\|\mathrm{Off}(\mathcal{V}_{\mathcal{J}\mathcal{J}})\| + \frac{13}{2}\frac{R_{\max}\lambda_{k+1}}{\widetilde{\lambda}_s^5}\mathrm{osc}(\mathcal{R}).$$

Our assumption implies that

$$\frac{\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s)}{\widetilde{\lambda}_s} \geqslant 2\|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{L}}\|$$

and hence, from (B.6), we get

$$s_{\min}(L_{\mathcal{J}}(\widetilde{\lambda}_s)) \geqslant \frac{1}{2}\frac{\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s)}{\widetilde{\lambda}_s}.$$

Next, we turn to the term on the right-hand side of (B.5). Applying the triangle inequality yields

$$\begin{aligned} &\|U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{I}} \cdot \Lambda_{\mathcal{I}}U_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s + U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U \cdot \Lambda U^{\mathrm{T}}\widetilde{u}_s\| \\ &\leqslant \|U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{I}}\| \cdot \|\Lambda_{\mathcal{I}}U_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s\| + \|U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\| \cdot \|\Lambda U^{\mathrm{T}}\widetilde{u}_s\|. \end{aligned}$$

For the bound on $\|\Lambda U^{\mathrm{T}}\widetilde{u}_s\|$, starting from $(A+E)\widetilde{u}_s = \widetilde{\lambda}_s\widetilde{u}_s$, we have $U\Lambda U^{\mathrm{T}}\widetilde{u}_s = (\widetilde{\lambda}_s I - E)\widetilde{u}_s$. Multiplying U^{T} from the left on both sides and taking the norms yield

$$\|\Lambda U^{\mathrm{T}}\widetilde{u}_s\| = \|U^{\mathrm{T}}(\widetilde{\lambda}_s I - E)\widetilde{u}_s\| \leqslant \widetilde{\lambda}_s + \|E\| \leqslant 2\widetilde{\lambda}_s.$$

The last step follows from (B.1) and our assumption $\lambda_s \geqslant 2\|E\|$. Similarly, we also have

$$\|\Lambda_{\mathcal{I}} U_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s\| \leqslant 2\widetilde{\lambda}_s.$$

The right-hand side of (B.5) is bounded by

$$2\widetilde{\lambda}_s\left(\|U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{I}}\| + \|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right).$$

It remains to estimate $\|U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{I}}\|$. We plug in the second-order expansion of $\Phi(\widetilde{\lambda}_s)$ in (A.5). By the orthogonality of eigenvectors, we get

$$\begin{aligned} U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{I}} &= \frac{U_{\mathcal{J}}^{\mathrm{T}}\mathcal{R}U_{\mathcal{I}}}{\widetilde{\lambda}_s^3} + U_{\mathcal{J}}^{\mathrm{T}}\varepsilon(\widetilde{\lambda}_s)U_{\mathcal{I}} \\ &= \frac{\mathcal{V}_{\mathcal{J}\mathcal{I}}}{\widetilde{\lambda}_s^3} + U_{\mathcal{J}}^{\mathrm{T}}\left(\varepsilon(\widetilde{\lambda}_s) - c^* I\right)U_{\mathcal{I}}, \end{aligned}$$

where $c^* = \frac{1}{2}\left(\max_i \varepsilon_i(\widetilde{\lambda}_s) + \min_i \varepsilon_i(\widetilde{\lambda}_s)\right)$. By Proposition 5.2, we obtain

$$\begin{aligned} \|U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{I}}\| &\leqslant \frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\widetilde{\lambda}_s^3} + \|\varepsilon(\widetilde{\lambda}_s) - c^* I\| \\ &\leqslant \frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\widetilde{\lambda}_s^3} + \frac{1}{2}\mathrm{osc}(\varepsilon(\widetilde{\lambda}_s)) \leqslant \frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\widetilde{\lambda}_s^3} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^5}\mathrm{osc}(\mathcal{R}). \end{aligned}$$

Therefore, the right-hand side of can be bounded by

$$2\widetilde{\lambda}_s\left(\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\widetilde{\lambda}_s^3} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^5}\mathrm{osc}(\mathcal{R}) + \|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right).$$

Finally, combining the above estimates, we continue from (B.5) to achieve the bound

$$\begin{aligned} \|U_{\mathcal{J}}^{\mathrm{T}}\widetilde{u}_s\| &\leqslant \frac{2\widetilde{\lambda}_s\left(\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\widetilde{\lambda}_s^3} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^5}\mathrm{osc}(\mathcal{R}) + \|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right)}{s_{\min}(L_{\mathcal{J}}(\widetilde{\lambda}_s))} \\ &\leqslant \frac{4}{\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s)}\left(\frac{\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\|}{\widetilde{\lambda}_s} + \frac{13}{2}\frac{R_{\max}}{\widetilde{\lambda}_s^3}\mathrm{osc}(\mathcal{R}) + \widetilde{\lambda}_s^2\|U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\|\right). \end{aligned}$$

Step 2. Bound on $\|U_{\mathcal{N}}^{\mathrm{T}}\widetilde{u}_s\|$. Denote the index set $\mathcal{K} := [r]\backslash\mathcal{N}$. We decompose

$$A = U_{\mathcal{N}}\Lambda_{\mathcal{N}}U_{\mathcal{N}}^{\mathrm{T}} + U_{\mathcal{K}}\Lambda_{\mathcal{K}}U_{\mathcal{K}}^{\mathrm{T}}.$$

Following the same derivation of (B.5), we have

$$\Big(I - U_{\mathcal{N}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{N}} \Lambda_{\mathcal{N}}\Big) U_{\mathcal{N}}^{\mathrm{T}} \widetilde{u}_s = U_{\mathcal{N}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{K}} \cdot \Lambda_{\mathcal{K}} U_{\mathcal{K}}^{\mathrm{T}} \widetilde{u}_s + U_{\mathcal{N}}^{\mathrm{T}} \Xi(\widetilde{\lambda}_s) U \cdot \Lambda U^{\mathrm{T}} \widetilde{u}_s. \quad \text{(B.7)}$$

Denote

$$\widehat{L}_{\mathcal{N}}(\widetilde{\lambda}_s) := I - U_{\mathcal{N}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{N}} \Lambda_{\mathcal{N}}.$$

We first bound $s_{\min}\big(\widehat{L}_{\mathcal{N}}(\widetilde{\lambda}_s)\big)$. Since $\Lambda_{\mathcal{N}}$ contains only negative eigenvalues, set $D_{\mathcal{N}} := \operatorname{diag}(|\lambda_l|)_{l \in \mathcal{N}} = -\Lambda_{\mathcal{N}} \succ 0$. Denote $Q_{\mathcal{N}} := U_{\mathcal{N}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{N}}$ for simplicity. Since $\phi_i(\widetilde{\lambda}_s) > 0$, $Q_{\mathcal{N}} \succ 0$. Rewrite

$$\widehat{L}_{\mathcal{N}}(\widetilde{\lambda}_s) = I + Q_{\mathcal{N}} D_{\mathcal{N}} = Q_{\mathcal{N}}^{1/2} \left(I + Q_{\mathcal{N}}^{1/2} D_{\mathcal{N}} Q_{\mathcal{N}}^{-1/2}\right) Q_{\mathcal{N}}^{1/2}.$$

Observe that $I + Q_{\mathcal{N}}^{1/2} D_{\mathcal{N}} Q_{\mathcal{N}}^{-1/2} \succeq I$. We get

$$s_{\min}\big(\widehat{L}_{\mathcal{N}}(\widetilde{\lambda}_s)\big) \geqslant \sqrt{\frac{\lambda_{\min}(Q_{\mathcal{N}})}{\lambda_{\max}(Q_{\mathcal{N}})}}.$$

To bound the eigenvalues of $Q_{\mathcal{N}}$, note that for any unit vector v, $v^{\mathrm{T}} Q_{\mathcal{N}} v = (U_{\mathcal{N}} v)^{\mathrm{T}} \Phi(\widetilde{\lambda}_s)(U_{\mathcal{N}} v)$. Combining Lemma 5.1 (i), we get

$$\frac{1}{2\widetilde{\lambda}_s} \leqslant \min_i \phi(\widetilde{\lambda}_s) \leqslant \lambda_{\min}(Q_{\mathcal{N}}) \leqslant \lambda_{\max}(Q_{\mathcal{N}}) \leqslant \max_i \phi(\widetilde{\lambda}_s) \leqslant \frac{3}{2\widetilde{\lambda}_s}$$

and thus

$$s_{\min}\big(\widehat{L}_{\mathcal{N}}(\widetilde{\lambda}_s)\big) \geqslant \sqrt{\frac{1}{3}}.$$

By the same estimation as in **Step 1**, we bound the right-hand side of (B.7) by

$$2\widetilde{\lambda}_s \left(\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\widetilde{\lambda}_s^3} + \frac{13}{2} \frac{R_{\max}}{\widetilde{\lambda}_s^5} \operatorname{osc}(\mathcal{R}) + \|U^{\mathrm{T}} \Xi(\widetilde{\lambda}_s) U\| \right).$$

Hence,

$$\|U_{\mathcal{N}}^{\mathrm{T}} \widetilde{u}_s\| \leqslant 2\sqrt{3} \left(\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\widetilde{\lambda}_s^2} + \frac{13}{2} \frac{R_{\max}}{\widetilde{\lambda}_s^4} \operatorname{osc}(\mathcal{R}) + \widetilde{\lambda}_s \|U^{\mathrm{T}} \Xi(\widetilde{\lambda}_s) U\| \right).$$

This completes the proof.

C Proof of Proposition 8.1

Before we prove Proposition 8.1, we introduce the necessary notation and a cumulant expansion formula to be used in the proof (adapted from [57]).

For any matrix $F \in \mathbb{R}^{n \times n}$ and vectors $v, w \in \mathbb{R}^n$, we write

$$F_{vw} := v^{\mathrm{T}} F w, \qquad F_{iw} := e_i^{\mathrm{T}} F w, \qquad F_{vj} := v^{\mathrm{T}} F e_j, \qquad F_{ij} := e_i^{\mathrm{T}} F e_j,$$

where $\{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{R}^n$. Moreover, for $1 \leqslant i, j \leqslant n$ define

$$\Delta^{ij} := (1 + \delta_{ij})^{-1}\big(e_i e_j^{\mathrm{T}} + e_j e_i^{\mathrm{T}}\big), \tag{C.1}$$

and for a differentiable function $f = f(H)$ of a symmetric matrix $H = (H_{ij})$ (with $\{H_{ij} : i \leqslant j\}$ treated as independent variables) set

$$\partial_{ij} f \ := \ \frac{\partial}{\partial H_{ij}} f(H) \ = \ \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f\big(H + t\,\Delta_{ij}\big).$$

Since there is potential ambiguity between $\big|\partial_{ij}^r(G_{vw})\big|$ and $\big|(\partial_{ij}^r G)_{vw}\big|$, we explicitly note that

$$\partial_{ij}^r\big(v^{\mathrm{T}} G w\big) \ = \ v^{\mathrm{T}}\big(\partial_{ij}^r G\big) w,$$

and throughout the proof we therefore write $\big|\partial_{ij}^r G_{vw}\big|$ with no confusion.

LEMMA C.1 (Cumulant expansion, Lemma 2.4 from [57])**.** *Let h be a real random variable with finite moments of all orders. For $k \geqslant 1$ define the k-th cumulant by*

$$\mathcal{C}_k(h) := (-\mathrm{i})^k \, \frac{\mathrm{d}^k}{\mathrm{d}t^k} \log \mathbb{E}\big[e^{\mathrm{i}th}\big]\bigg|_{t=0}.$$

Let $f : \mathbb{R} \to \mathbb{C}$ be smooth, and write $f^{(k)}$ for its k-th derivative. Fix $\ell \in \mathbb{N}$. Then

$$\mathbb{E}\big[hf(h)\big] = \sum_{k=0}^{\ell} \frac{1}{k!}\, \mathcal{C}_{k+1}(h)\, \mathbb{E}\big[f^{(k)}(h)\big] + \mathtt{R}_{\ell+1},$$

where the remainder $R_{\ell+1}$ satisfies, for any $t > 0$,

$$\begin{aligned} |\mathtt{R}_{\ell+1}| \leqslant 2^{\ell}(\ell+2) \cdot \left(\mathbb{E}\Big[\sup_{|x| \leqslant |h|} \big|f^{(\ell+1)}(x)\big|^2\Big] \cdot \mathbb{E}\big[|h|^{2\ell+4} \mathbf{1}(|h| > t)\big] \right)^{1/2} \\ + 2^{\ell}(\ell+2) \cdot \mathbb{E}|h|^{\ell+2} \cdot \sup_{|x| \leqslant t} \big|f^{(\ell+1)}(x)\big|. \end{aligned}$$

Now we provide the proof. Define

$$L := \operatorname{diag}(l_1, \ldots, l_n) \quad \text{with} \quad l_k := \sum_{s=1}^{n} \sigma_{ks}^2 G_{ss}. \tag{C.2}$$

The main technical task is to establish:

PROPOSITION C.1. *Fix $D > 0$. Let $v, w \in \mathbb{R}^n$ be deterministic unit vectors, and let $z \in \mathbb{C}$ satisfy $|z| \geqslant 6\sqrt{R_{\max}}$.*

(i) *Under the general sub-Gaussian regime, with probability at least $1 - 6n^{-D-4}$,*

$$\big|v^{\mathrm{T}}(E - L)Gw\big| \leqslant C(D+6)^{3/2} \frac{\beta K \sigma_{\max} \log^2 n}{|z|} + Cn^{-500},$$

where $C \geqslant 1$ is an absolute constant.

(ii) *Under the bounded sparse-entry regime, namely if*

$$\max_{i,j} |E_{ij}| \leqslant c_0 K\sigma_{\max} \quad \textit{almost surely} \qquad \textit{and} \qquad \beta \leqslant \beta_0,$$

then, with probability at least $1 - 6n^{-D-4}$,

$$\left|v^{\mathrm{T}}(E-L)Gw\right| \leqslant C(D+6)(c_0+\beta_0)\frac{K\sigma_{\max}\log n}{|z|},$$

where $C > 0$ *is an absolute constant.*

The key to proving Proposition C.1 is the following moment estimates, whose proof is deferred to Section C.1. For brevity, we denote

$$\mathcal{D} := (E-L)G.$$

and bound the moments of

$$\mathcal{D}_{vw} = v^{\mathrm{T}}(E-L)Gw.$$

We define a regime-dependent truncation parameter

$$\mathtt{L}_n := \begin{cases} 100\sqrt{D+6}K\sigma_{\max}\log n, & \text{in the general sub-Gaussian regime,} \\ c_0 K\sigma_{\max}, & \text{in the bounded sparse-entry regime.} \end{cases} \tag{C.3}$$

Define

$$\Omega_n^0 := \left\{ \|E\| \leqslant 3\sqrt{R_{\max}}, \quad \max_{a,b} |E_{ab}| \leqslant \mathtt{L}_n \right\}$$

and a slightly larger event

$$\Omega_n := \left\{ \|E\| \leqslant 3.5\sqrt{R_{\max}}, \quad \max_{a,b} |E_{ab}| \leqslant 2\mathtt{L}_n \right\}. \tag{C.4}$$

Let $\chi = \chi(E) \in [0,1]$ be a smooth cutoff such that

$$\chi(E) = 1 \quad \text{on } \Omega_n^0, \qquad \chi(E) = 0 \quad \text{outside } \Omega_n.$$

Our main technical estimate is the following bound:

LEMMA C.2. *For any integer* $p \geqslant 3$,

$$\begin{aligned} \mathbb{E}(|\mathcal{D}_{vw}|^{2p}\chi(E)) \leqslant & CK\beta \sum_{m=1}^{3} p^{m-1}\frac{\sigma_{\max}^m}{|z|^m}\mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m}\chi(E)\right] \\ & + CK^2\beta^2 \sum_{m=1}^{4} p^{m-1}\frac{\sigma_{\max}^{m+1}}{|z|^{m+1}}\mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m}\chi(E)\right] + \varepsilon_p \end{aligned}$$

for an absolute constant $C \geqslant 1$. *Here, we define*

$$\varepsilon_p := C^p p^3 \exp\left(-5000(D+6)(\log n)^2\right)\frac{(\beta K\sigma_{\max})^2}{|z|^2} \tag{C.5}$$

in the general sub-Gaussian regime and $\varepsilon_p := 0$ *in the bounded sparse-entry regime.*

Using this lemma, we first obtain a bound on

$$\mathtt{Z}_p := \left(\mathbb{E}(|\mathcal{D}_{vw}|^{2p}\chi(E))\right)^{1/2p}.$$

By the Hölder inequality, for any $q \leqslant 2p$,

$$\mathbb{E}[|\mathcal{D}_{vw}|^q\chi(E)] \leqslant \left(\mathbb{E}|\mathcal{D}_{vw}|^{2p}\chi(E)\right)^{\frac{q}{2p}} = \mathtt{Z}_p^q.$$

The same argument gives

$$\mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_n}{|z|}\right)^q \chi(E)\right] \leqslant \left(\mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_n}{|z|}\right)^{2p} \chi(E)\right]\right)^{\frac{q}{2p}}.$$

Note that by the Minkowski inequality,

$$\left(\mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_\mathtt{n}}{|z|}\right)^{2p} \chi(E)\right]\right)^{\frac{1}{2p}} \leqslant \left(\mathbb{E}|\mathcal{D}_{vw}|^{2p}\chi(E)\right)^{\frac{1}{2p}} + 22\frac{\mathtt{L}_\mathtt{n}}{|z|} = \mathtt{Z}_p + 22\frac{\mathtt{L}_\mathtt{n}}{|z|}.$$

We get

$$\mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_\mathtt{n}}{|z|}\right)^q \chi(E)\right] \leqslant \left(\mathtt{Z}_p + 22\frac{\mathtt{L}_n}{|z|}\right)^q.$$

Therefore, Lemma C.2 implies

$$\begin{aligned}\mathtt{Z}_p^{2p} \leqslant & CK\beta\sum_{m=1}^{3} p^{m-1}\frac{\sigma_{\max}^m}{|z|^m}\left(\mathtt{Z}_p + 22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m} \\ & + CK^2\beta^2\sum_{m=1}^{4} p^{m-1}\frac{\sigma_{\max}^{m+1}}{|z|^{m+1}}\left(\mathtt{Z}_p + 22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m} + \varepsilon_p,\end{aligned} \tag{C.6}$$

where ε_p is defined in (C.5).

We claim that

$$\mathtt{Z}_p \leqslant C\left(p\frac{\mathtt{L}_\mathtt{n}}{|z|} + p\frac{\beta K\sigma_{\max}}{|z|} + \left(\frac{\beta K\sigma_{\max}}{|z|}\right)^2 + \varepsilon_p^{1/(2p)}\right). \tag{C.7}$$

To see this, consider the case where $\mathtt{Z}_p \leqslant 88p\frac{\mathtt{L}_\mathtt{n}}{|z|} + 2\varepsilon_p^{1/(2p)}$. In this regime, (C.7) holds trivially. Now, let us assume $\mathtt{Z}_p > 88p\frac{\mathtt{L}_\mathtt{n}}{|z|}$ and $\mathtt{Z}_p > 2\varepsilon_p^{1/(2p)}$. Thus, $\mathtt{Z}_p + 22\frac{\mathtt{L}_n}{|z|} \leqslant (1+\frac{1}{4p})\mathtt{Z}_p$ and for $1 \leqslant m \leqslant 4$,

$$\left(\mathtt{Z}_p + 22\frac{\mathtt{L}_\mathtt{n}}{|z|}\right)^{2p-m} \leqslant \left(1 + \frac{1}{4p}\right)^{2p}\mathtt{Z}_p^{2p-m} \leqslant e^{1/2}\mathtt{Z}_p^{2p-m}.$$

Moreover, the last term in (C.6) satisfies $\varepsilon_p \leqslant 2^{-2p}\mathtt{Z}_p^{2p}$ and can be absorbed into the left-hand side. Hence, (C.6) simplifies to

$$\mathtt{Z}_p^{2p} \leqslant \sum_{m=1}^{3} CK\beta p^{m-1}\frac{\sigma_{\max}^m}{|z|^m}\mathtt{Z}_p^{2p-m} + \sum_{m=1}^{4} CK^2\beta^2 p^{m-1}\frac{\sigma_{\max}^{m+1}}{|z|^{m+1}}\mathtt{Z}_p^{2p-m} \tag{C.8}$$

for an absolute constant $C \geqslant 1$. Next, we apply Young's inequality for $\mathtt{B} \cdot \mathtt{Z}_p^{2p-m}$. Let $\delta > 0$ be a small constant to be specified later. Since

$$\mathtt{B} \cdot \mathtt{Z}_p^{2p-m} = (\delta \mathtt{Z}_p^{2p-m}) \cdot \frac{\mathtt{B}}{\delta} \leqslant \frac{(\delta \mathtt{Z}_p^{2p-m})^{2p/(2p-m)}}{2p/(2p-m)} + \frac{(\mathtt{B}/\delta)^{2p/m}}{2p/m}$$
$$= \frac{\delta^{2p/(2p-m)}}{2p/(2p-m)} \mathtt{Z}_p^{2p} + \frac{(1/\delta)^{2p/m}}{2p/m} \mathtt{B}^{\frac{2p}{m}}.$$

Choose $\delta = 1/100$. Then $\frac{\delta^{2p/(2p-m)}}{2p/(2p-m)} < 1/10$ for $1 \leqslant m \leqslant 4$. We absorb the $\mathtt{Z}_p^{2p}$ terms to the left-hand side of (C.8). Continuing from (C.8), we have

$$\mathtt{Z}_p^{2p} \leqslant \sum_{m=1}^{3} \left(CK\beta p^{m-1} \frac{\sigma_{\max}^m}{|z|^m} \right)^{\frac{2p}{m}} + \sum_{m=1}^{4} \left(CK^2\beta^2 p^{m-1} \frac{\sigma_{\max}^{m+1}}{|z|^{m+1}} \right)^{\frac{2p}{m}}.$$

Taking the $2p$-th roots on both sides yields

$$\mathtt{Z}_p \leqslant C \max_{1 \leqslant m \leqslant 3} \left(K\beta p^{m-1} \frac{\sigma_{\max}^m}{|z|^m} \right)^{\frac{1}{m}} + C \max_{1 \leqslant m \leqslant 4} \left(K^2\beta^2 p^{m-1} \frac{\sigma_{\max}^{m+1}}{|z|^{m+1}} \right)^{\frac{1}{m}}.$$

Since $K\beta \geqslant 1$, $p \geqslant 3$, and $|z| \geqslant \sigma_{\max}$, we have for $1 \leqslant m \leqslant 3$,

$$\left(K\beta p^{m-1} \frac{\sigma_{\max}^m}{|z|^m} \right)^{\frac{1}{m}} = (K\beta)^{\frac{1}{m}} p^{\frac{m-1}{m}} \frac{\sigma_{\max}}{|z|} \leqslant p \frac{\beta K \sigma_{\max}}{|z|}$$

and for $2 \leqslant m \leqslant 4$,

$$(K\beta)^{\frac{2}{m}} p^{\frac{m-1}{m}} \left(\frac{\sigma_{\max}}{|z|} \right)^{1+\frac{1}{m}} = \left(\frac{\beta K \sigma_{\max}}{|z|} \right) \cdot (K\beta)^{\frac{2-m}{m}} \cdot p^{\frac{m-1}{m}} \cdot \left(\frac{\sigma_{\max}}{|z|} \right)^{\frac{1}{m}}$$
$$\leqslant p \frac{\beta K \sigma_{\max}}{|z|}.$$

The above inequality simplifies to

$$\mathtt{Z}_p \leqslant C \left(p \frac{\beta K \sigma_{\max}}{|z|} + \left(\frac{\beta K \sigma_{\max}}{|z|} \right)^2 \right).$$

Combining this bound with the exceptional case proves (C.7).

Now we prove Proposition 8.1. Let us first consider the general sub-Gaussian regime. By the union bound and our sub-Gaussian tail assumption,

$$\mathbb{P}\left(\max_{a,b} |E_{ab}| > 2\mathtt{L}_n \right) \leqslant \sum_{a,b} \mathbb{P}(|E_{ab}| > \mathtt{L}_n) \leqslant 2n^2 \exp\left(-10^4 (D+6)(\log n)^2 \right) \leqslant 2n^{-D-4}.$$

Moreover, by Lemma 5.4, we have

$$\mathbb{P}(\|E\| > 3\sqrt{R_{\max}}) \leqslant 3n^{-D-4}$$

by selecting the constant C_0 in Assumption 1.2 sufficiently large. Hence,

$$\mathbb{P}\big((\Omega_n^0)^c\big) \leqslant 5n^{-D-4}.$$

From the decomposition $|\mathcal{D}_{vw}| = |\mathcal{D}_{vw}|\mathbf{1}_{\Omega_n^0} + |\mathcal{D}_{vw}|\mathbf{1}_{(\Omega_n^0)^c}$ and $\chi(E) = 1$ on Ω_n^0, we have $|\mathcal{D}_{vw}|\mathbf{1}_{\Omega_n^0} \leqslant |\mathcal{D}_{vw}|\chi(E)$. Thus

$$|\mathcal{D}_{vw}| \leqslant |\mathcal{D}_{vw}|\chi(E) + |\mathcal{D}_{vw}|\mathbf{1}_{(\Omega_n^0)^c}.$$

It follows that

$$\mathbb{P}(|\mathcal{D}_{vw}| \geqslant t) \leqslant \mathbb{P}(|\mathcal{D}_{vw}|\chi(E) \geqslant t) + \mathbb{P}\big((\Omega_n^0)^c\big).$$

Then by Markov's inequality and $0 \leqslant \chi \leqslant 1$, for any integer $p \geqslant 3$,

$$\begin{aligned}\mathbb{P}(|\mathcal{D}_{vw}| \geqslant t) &\leqslant \frac{\mathbb{E}(|\mathcal{D}_{vw}|^{2p}\chi(E))}{t^{2p}} + 5n^{-D-4}\\ &= \left(\frac{\mathsf{Z}_p}{t}\right)^{2p} + 5n^{-D-4}.\end{aligned}$$

We choose $p = \lceil (D+4)\log n \rceil$ and in view of (C.7),

$$t = e \cdot C\left(p\frac{\mathsf{L}_n}{|z|} + p\frac{\beta K\sigma_{\max}}{|z|} + \left(\frac{\beta K\sigma_{\max}}{|z|}\right)^2 + \varepsilon_p^{1/(2p)}\right).$$

With this choice of t,

$$\left(\frac{\mathsf{Z}_p}{t}\right)^{2p} \leqslant e^{-2p} \leqslant \exp\big(-2(D+4)\log n\big) = n^{-2D-8} \leqslant n^{-D-4}.$$

It follows that

$$\mathbb{P}(|\mathcal{D}_{vw}| \geqslant t) \leqslant n^{-D-4} + 5n^{-D-4} = 6n^{-D-4}.$$

Finally, let us simplify the terms in t under the choice of p. For the term $\varepsilon_p^{1/(2p)}$, note that by Assumption 1.2, $R_{\max} \leqslant n\sigma_{\max}^2$ and $\sqrt{R_{\max}} \geqslant C_0(D+8)K\sigma_{\max}\log n$. We have

$$K \leqslant \frac{\sqrt{n}}{C_0(D+8)\log n}.$$

As discussed in Remark 1.1, $\beta \leqslant C\sqrt{\log(eK)} \leqslant C\sqrt{\log n}$. Under our assumption $|z| \geqslant 6\sqrt{R_{\max}}$, $\frac{\beta K\sigma_{\max}}{|z|} \leqslant C'$. Thus

$$\begin{aligned}\varepsilon_p^{1/(2p)} &= C^{1/2}p^{3/(2p)}\exp\left(-\frac{10(D+6)\log^2 n}{2p}\right)\left(\frac{\beta K\sigma_{\max}}{|z|}\right)^{1/p}\\ &\leqslant C\exp\left(-\frac{5000(D+6)\log^2 n}{2(D+5)\log n}\right) \leqslant C\exp(-500\log n) = Cn^{-500}.\end{aligned}$$

Next, substituting $p \leqslant (D+5)\log n$ and $\mathsf{L}_n = 5\sqrt{D+6}K\sigma_{\max}\log n$, we also have

$$p\frac{\mathsf{L}_n}{|z|} \leqslant 5(D+6)^{3/2}\frac{K\sigma_{\max}\log^2 n}{|z|} \quad \text{and} \quad p\frac{\beta K\sigma_{\max}}{|z|} \leqslant (D+5)\frac{\beta K\sigma_{\max}\log n}{|z|}.$$

Note that

$$p\frac{\beta K\sigma_{\max}}{|z|} \geqslant \left(\frac{\beta K\sigma_{\max}}{|z|}\right)^2 \tag{C.9}$$

by selecting the constant C_0 sufficiently large in Assumption 1.2. Thus, we conclude that with probability at least $1-6n^{-D-4}$,

$$\begin{aligned}|\mathcal{D}_{vw}| &\leqslant C\left(5(D+6)^{3/2}\frac{K\sigma_{\max}\log^2 n}{|z|} + (D+5)\frac{\beta K\sigma_{\max}\log n}{|z|} + n^{-500}\right)\\ &\leqslant C(D+6)^{3/2}\frac{K\sigma_{\max}\log^2 n}{|z|} + Cn^{-500}.\end{aligned}$$

We now prove the bounded sparse-entry case. In this regime, $\mathtt{L}_n = c_0 K\sigma_{\max}$ and $\varepsilon_p = 0$. Thus from (C.7), together with (C.9), we obtain

$$\mathtt{Z}_p \leqslant C\left(p\frac{c_0 K\sigma_{\max}}{|z|} + p\frac{\beta K\sigma_{\max}}{|z|}\right).$$

Moreover, the entrywise truncation event is deterministic, i.e., $\max_{i,j}|E_{ij}| \leqslant \mathtt{L}_n$ almost surely. Taking $p = \lceil (D+4)\log n\rceil$ and applying the same Markov argument as above, together with the operator norm event, yields

$$|\mathcal{D}_{vw}| \leqslant C(c_0+\beta_0)(D+6)\frac{K\sigma_{\max}\log n}{|z|}$$

with probability at least $1-6n^{-D-4}$. This proves part (ii) and completes the proof of Proposition C.1.

C.1 Proof of Lemma C.2

We prove Lemma C.2 in this section. Let

$$Q := \mathcal{D}_{vw}^{p-1}\overline{\mathcal{D}}_{vw}^{p}.$$

We rewrite

$$\mathbb{E}[|\mathcal{D}_{vw}|^{2p}\chi(E)] = \mathbb{E}\left[(EG)_{vw}Q\chi(E)\right] - \mathbb{E}\left[(LG)_{vw}\,Q\chi(E)\right]. \tag{C.10}$$

We analyze the first term on the right-hand side of (C.10) using the cumulant expansion. For $1 \leqslant i,j \leqslant n$, define the auxiliary functions

$$h_{ij}(E) := G_{jw}Q, \qquad f_{ij}^{\chi}(E) := h_{ij}(E)\chi(E).$$

Thus

$$\mathbb{E}\left[(EG)_{vw}Q\,\chi(E)\right] = \sum_{i,j} v_i\,\mathbb{E}\left[E_{ij}f_{ij}^{\chi}(E)\right].$$

Let $\chi \equiv \chi(E)$. For $l \leqslant 3$, the product rule gives

$$\partial_{ij}^l f_{ij}^{\chi} = \chi\cdot\partial_{ij}^l h_{ij} + \sum_{a=1}^{l}\binom{l}{a}(\partial_{ij}^{l-a}h_{ij})(\partial_{ij}^a\chi).$$

The first term on the right-hand side is the main term estimated below. The remaining terms contain at least one derivative of the cutoff function. We choose χ so that, for $1 \leqslant a \leqslant 3$,

$$|\partial_{ij}^{a}\chi(E)| \leqslant C_a \mathtt{L}_n^{-a}.$$

These derivative terms are supported in $\Omega_n \backslash \Omega_n^0$. Using the derivative bounds proved below, together with Cauchy-Schwarz inequality and the high-probability estimate for $(\Omega_n^0)^c$, the total contribution of the terms containing $\partial_{ij}^a\chi$, $a \geqslant 1$, is absorbed into an error $O(n^{-D-5})$, after increasing C_0 if necessary.

In the estimates below, we write only the terms in which no derivative falls on χ. The terms involving derivatives of χ are supported on $\Omega_n \backslash \Omega_n^0$. By the same derivative estimates used below and the probability bound for this transition region, their total contribution is negligible compared with the final high-probability bound. We therefore record only the main terms in the moment estimate. The remainder ε_p below refers only to the tail part of the cumulant remainder; in the bounded sparse-entry regime this tail part is zero. Therefore, to simplify notation, we suppress the cutoff $\chi(E)$ from the main estimates. From this point on, we simply work with

$$f_{ij}(E) := G_{jw}Q$$

Since E is symmetric, the independent random variables are E_{ij} $(1 \leqslant i \leqslant j \leqslant n)$. For $i < j$, we write ∂_{ij} for differentiation with respect to $E_{ij} = E_{ji}$, and we set $\partial_{ji} := \partial_{ij}$. For the diagonal entries, ∂_{ii} denotes differentiation with respect to E_{ii}. With this convention, for $x, y \in \mathbb{R}^n$,

$$\partial_{ij}G_{xy} = (1+\delta_{ij})^{-1}(G_{xi}G_{jy} + G_{xj}G_{iy}), \tag{C.11}$$

Now, from

$$(EG)_{vw}Q = \sum_i v_i E_{ii} G_{iw} Q + \sum_{i<j} E_{ij}\big(v_i G_{jw} + v_j G_{iw}\big)Q = \sum_{i,j} v_i E_{ij} f_{ij}(E),$$

we apply the cumulant expansion Lemma C.1 to each independent entry E_{ij} $(1 \leqslant i \leqslant j \leqslant n)$ with $l = 2$ to get

$$\mathbb{E}[(EG)_{vw}Q] := X_1 + X_2 + \mathtt{R}_3,$$

where for $k = 1, 2$,

$$X_k := \sum_{i,j} v_i \frac{1}{k!}\mathcal{C}_{k+1}(E_{ij})\mathbb{E}\left[\partial_{ij}^k f_{ij}(E)\right]. \tag{C.12}$$

Here the $k = 0$ cumulant term vanishes because the entries of E are centered. The remainder is written as

$$\mathtt{R}_3 := \sum_{i,j} v_i \mathtt{R}_3^{ij} \tag{C.13}$$

where

$$\mathtt{R}_3^{ij} := \mathbb{E}\big[E_{ij} f_{ij}(E)\big] - \sum_{k=1}^{2} \frac{1}{k!}\mathcal{C}_{k+1}(E_{ij})\mathbb{E}\big[\partial_{ij}^k f_{ij}(E)\big].$$

With these notations, we have

$$\mathbb{E}[|\mathcal{D}_{vw}|^{2p}] = (X_1 - \mathbb{E}\left[(LG)_{vw}Q\right]) + X_2 + \mathtt{R}_3. \tag{C.14}$$

The next lemma bounds each term on the right-hand side of (C.14).

LEMMA C.3. *(i) For X_1,*

$$|X_1 - \mathbb{E}\left[(LG)_{vw}\, Q\right]| \leqslant \frac{2}{3}\sigma_{\max}\frac{\mathbb{E}\left[|\mathcal{D}_{vw}|^{2p-1}\right]}{|z|} + 36p\sigma_{\max}^2\frac{\mathbb{E}[|\mathcal{D}_{vw}|^{2p-2}]}{|z|^2}.$$

(ii) For X_2,

$$|X_2| \leqslant \frac{4}{9}\beta K\sigma_{\max}\frac{\mathbb{E}\left[|\mathcal{D}_{vw}|^{2p-1}\right]}{|z|}+178p\beta K\sigma_{\max}^2\frac{\mathbb{E}[|\mathcal{D}_{vw}|^{2p-2}]}{|z|^2}+700p^2\beta K\sigma_{\max}^3\frac{\mathbb{E}[|\mathcal{D}_{vw}|^{2p-3}]}{|z|^3}.$$

(iii) For $\mathtt{R}_3$, recalling $\mathtt{L}_n$ from (C.3)*, we have*

$$\begin{aligned}|\mathtt{R}_3| \leqslant& CK^2\beta^2\sum_{m=1}^{4}p^{m-1}\frac{\sigma_{\max}^{m+1}}{|z|^{m+1}}\mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m}\right]\\ &+ C^p p^3\exp\big(-10(D+6)\log^2 n\big)\frac{(\beta K\sigma_{\max})^2}{|z|^2}\end{aligned}$$

for an absolute constant $C \geqslant 1$.

Again, we emphasize that in Lemma C.3, we suppress $\chi(E)$ from the notation. More precisely, every expectation in the bounds for X_1, X_2, and $\mathtt{R}_3$ is understood to include the factor $\chi(E)$, except for the explicitly separated cutoff-derivative terms.

With this lemma, the conclusion of Lemma C.2 follows from (C.14):

$$\begin{aligned}\mathbb{E}\big[|\mathcal{D}_{vw}|^{2p}\chi(E)\big] &\leqslant |X_1 - \mathbb{E}\left[(LG)_{vw}\, Q\right]| + |X_2| + |\mathtt{R}_3|\\ &\leqslant CK\beta\sum_{m=1}^{3}p^{m-1}\frac{\sigma_{\max}^m}{|z|^m}\mathbb{E}\left[|\mathcal{D}_{vw}|^{2p-m}\chi(E)\right]\\ &\quad + CK^2\beta^2\sum_{m=1}^{4}p^{m-1}\frac{\sigma_{\max}^{m+1}}{|z|^{m+1}}\mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m}\chi(E)\right]\\ &\quad + C^p p^3\exp\big(-10(D+6)\log^2 n\big)\frac{(\beta K\sigma_{\max})^2}{|z|^2}.\end{aligned}$$

In the remainder of this section, we prove Lemma C.3.

Proof of Lemma C.3 (i). From the definition of $L = \operatorname{diag}(\sum_{j=1}^n \sigma_{kj}^2 G_{jj})_{1\leqslant k\leqslant n}$ in (C.2), we have

$$\mathbb{E}[(LG)_{vw}Q] = \mathbb{E}[\sum_{i,j} v_i L_{ij} G_{jw} Q] = \sum_{i,j} v_i\sigma_{ij}^2\mathbb{E}[G_{jj}G_{iw}Q]. \tag{C.15}$$

From (C.12), by product rule, we get

$$\begin{aligned}X_1 &= \sum_{i,j} v_i\mathcal{C}_2(E_{ij})\mathbb{E}[\partial_{ij}f_{ij}(E)]\\ &= \sum_{i,j} v_i\sigma_{ij}^2\mathbb{E}[\partial_{ij}(G_{jw}Q)]\\ &= \sum_{i,j} v_i\sigma_{ij}^2\mathbb{E}\left[(\partial_{ij}G_{jw})Q\right] + \sum_{i,j} v_i\sigma_{ij}^2\mathbb{E}\left[G_{jw}(\partial_{ij}Q)\right].\end{aligned}$$

Plugging in $\partial_{ij}G_{jw} = (1+\delta_{ij})^{-1}\big(G_{ji}G_{jw} + G_{jj}G_{iw}\big)$ by (C.11), we further obtain

$$X_1 = \sum_{i,j} v_i\sigma_{ij}^2\mathbb{E}\left[G_{jj}G_{iw}Q\right] + \sum_{i\neq j} v_i\sigma_{ij}^2\mathbb{E}\left[G_{ji}G_{jw}Q\right] + \sum_{i,j} v_i\sigma_{ij}^2\mathbb{E}\left[G_{jw}(\partial_{ij}Q)\right].$$

Comparing with (C.15), we arrive at

$$X_1 - \mathbb{E}\left[(LG)_{vw}\,Q\right] = T^{(1)} + T^{(2)},$$

where

$$T^{(1)} := \mathbb{E}\Big[\sum_{i\neq j} v_i\sigma_{ij}^2 G_{ji}G_{jw}Q\Big], \qquad T^{(2)} := \mathbb{E}\Big[\sum_{i,j} v_i\sigma_{ij}^2 G_{jw}(\partial_{ij}Q)\Big].$$

We now provide upper bounds for $T^{(1)}$ and $T^{(2)}$.

Step 1. *Estimates for* $T^{(1)}$. Applying Cauchy-Schwarz, we have

$$\Big|\sum_{i\neq j} v_i\sigma_{ij}^2 G_{ji}G_{jw}\Big| \leqslant \Big(\sum_{i\neq j} v_i^2\sigma_{ij}^2|G_{ij}|^2\Big)^{1/2}\Big(\sum_{i\neq j}\sigma_{ij}^2|G_{jw}|^2\Big)^{1/2}.$$

Note that

$$\sum_{i\neq j} v_i^2\sigma_{ij}^2|G_{ij}|^2 \leqslant \sum_i v_i^2\big(\sum_j \sigma_{ij}^2|G_{ij}|^2\big) \leqslant \sigma_{\max}^2\sum_i v_i^2\|Ge_i\|^2 \leqslant \sigma_{\max}^2\|G\|^2$$

and

$$\sum_{i\neq j}\sigma_{ij}^2|G_{jw}|^2 \leqslant \sum_j |G_{jw}|^2\big(\sum_i\sigma_{ij}^2\big) \leqslant R_{\max}\|Gw\|^2 \leqslant R_{\max}\|G\|^2.$$

We get

$$\Big|\sum_{i,j} v_i\sigma_{ij}^2 G_{ji}G_{jw}\Big| \leqslant \sigma_{\max}\sqrt{R_{\max}}\|G\|^2 \leqslant \frac{4\,\sigma_{\max}\sqrt{R_{\max}}}{|z|^2} \leqslant \frac{2}{3}\frac{\sigma_{\max}}{|z|}.$$

by applying (5.6) and the assumption $|z| \geqslant 6\sqrt{R_{\max}}$. Consequently,

$$|T^{(1)}| \leqslant \frac{2}{3}\frac{\sigma_{\max}}{|z|}\mathbb{E}\left[|\mathcal{D}_{vw}|^{2p-1}\right].$$

Step 2. *Estimates for* $T^{(2)}$. Recall that

$$T^{(2)} := \sum_{i,j} v_i\sigma_{ij}^2\mathbb{E}\left[G_{jw}\partial_{ij}(\mathcal{D}_{vw}^{p-1}\overline{\mathcal{D}}_{vw}^{p})\right].$$

For $p \geqslant 2$, by the product rule,

$$\partial_{ij}(\mathcal{D}_{vw}^{p-1}\overline{\mathcal{D}}_{vw}^{p}) = (p-1)(\partial_{ij}\mathcal{D}_{vw})\mathcal{D}_{vw}^{p-2}\overline{\mathcal{D}}_{vw}^{p} + p(\partial_{ij}\overline{\mathcal{D}}_{vw})|\mathcal{D}_{vw}|^{2(p-1)}.$$

For $p=1$, the first term is absent, and the same final bound follows. Hence, it suffices to bound the two quantities

$$\mathtt{A} := \sum_{i,j} v_i \sigma_{ij}^2 G_{jw}(\partial_{ij}\mathcal{D}_{vw}), \qquad \bar{\mathtt{A}} := \sum_{i,j} v_i \sigma_{ij}^2 G_{jw}(\partial_{ij}\overline{\mathcal{D}}_{vw}).$$

We first estimate $\mathtt{A}$. From the identity $(zI-E)G=I$, we have $EG=zG-I$. Plugging into $\mathcal{D}=(E-L)G=EG-LG$ yields

$$\mathcal{D}=(zI-L)G-I.$$

Thus $\partial_{ij}\mathcal{D}=(zI-L)\partial_{ij}G-(\partial_{ij}L)G$. Recall Δ_{ij} from (C.1). We also have

$$\partial_{ij}\mathcal{D}_{vw}=v^{\mathrm{T}}(zI-L)G\Delta^{ij}Gw-v^{\mathrm{T}}(\partial_{ij}L)Gw.$$

Denote $\widetilde{v}^{\mathrm{T}} := v^{\mathrm{T}}(zI-L)$. From the definition of L in (C.2) and the partial derivative formula (C.11), we compute

$$\partial_{ij} l_s = \sum_k \sigma_{sk}^2 \partial_{ij} G_{kk} = 2(1+\delta_{ij})^{-1}\sum_k \sigma_{sk}^2 G_{ki}G_{kj}.$$

Hence,

$$v^{\mathrm{T}}(\partial_{ij}L)Gw = 2(1+\delta_{ij})^{-1}\sum_{s,k} v_s\sigma_{s,k}^2 G_{ki}G_{kj}G_{sw}.$$

It follows that

$$\partial_{ij}\mathcal{D}_{vw}=(1+\delta_{ij})^{-1}\Big(G_{\widetilde{v}i}G_{jw}+G_{\widetilde{v}j}G_{iw}-2\sum_{s,k}v_s\sigma_{sk}^2G_{ki}G_{kj}G_{sw}\Big). \tag{C.16}$$

We are ready to bound

$$\begin{aligned}|\mathtt{A}| = \Big|\sum_{i,j} v_i\sigma_{ij}^2 G_{jw}(\partial_{ij}\mathcal{D}_{vw})\Big| &\leqslant \Big|\sum_{i,j} v_i\sigma_{ij}^2 G_{jw}^2 G_{\widetilde{v}i}\Big| + \Big|\sum_{i,j} v_i\sigma_{ij}^2 G_{jw}G_{\widetilde{v}j}G_{iw}\Big| \\ &\quad + 2\Big|\sum_{i,j} v_i\sigma_{ij}^2 G_{jw}\sum_{s,k} v_s\sigma_{s,k}^2 G_{ki}G_{kj}G_{sw}\Big| \\ &:= T_1^{(2)}+T_2^{(2)}+T_3^{(2)}.\end{aligned}$$

Note that

$$\|L\| \leqslant R_{\max}\|G\| \leqslant 2\frac{R_{\max}}{|z|} \leqslant \frac{1}{18}|z|$$

by (5.6) and our supposition $|z|>6\sqrt{R_{\max}}$. Thus $\|\widetilde{v}\| \leqslant \|zI-L\| \leqslant |z|+\|L\| \leqslant \frac{19}{18}|z|$. We will repeatedly use $\sum_i |G_{iw}|^2 = \|Gw\|^2 \leqslant \|G\|^2$.

Now we estimate $T_1^{(2)}, T_2^{(2)}$ and $T_3^{(2)}$ respectively. For $T_1^{(2)}$, since $\max_{i,j}\sigma_{ij}^2 \leqslant \sigma_{\max}^2$,

$$\begin{aligned}T_1^{(2)} &\leqslant \sigma_{\max}^2 \sum_i |v_i||G_{\widetilde{v}i}| \cdot \sum_j |G_{jw}|^2 \leqslant \sigma_{\max}^2\|Gw\|^2\sqrt{\sum_i |v_i|^2 \sum_i |G_{\widetilde{v}i}|^2} \\ &\leqslant \sigma_{\max}^2\|G\|^2\|\widetilde{v}^{\mathrm{T}}G\| \leqslant \sigma_{\max}^2\|G\|^3\|\widetilde{v}\| \leqslant \frac{76\sigma_{\max}^2}{9|z|^2}.\end{aligned}$$

Similarly,

$$T_2^{(2)} \leqslant \sigma_{\max}^2 \sum_i |v_i||G_{iw}| \cdot \sum_j |G_{jw}||G_{\widetilde{v}j}| \leqslant \sigma_{\max}^2 \sqrt{\sum_i |v_i|^2 \sum_i |G_{iw}|^2} \sqrt{\sum_j |G_{jw}|^2 \sum_j |G_{\widetilde{v}j}|^2}$$
$$\leqslant \sigma_{\max}^2 \|G\|^3 \|\widetilde{v}\| \leqslant \frac{76\sigma_{\max}^2}{9|z|^2}.$$

For $T_3^{(2)}$, observe that

$$\sum_s |v_s \sigma_{sk}^2||G_{sw}| \leqslant \sigma_{\max}^2 \sum_s |v_s||G_{sw}| \leqslant \sigma_{\max}^2 \|G\|$$

and

$$\sum_k |G_{ki}G_{kj}| \leqslant \sqrt{\sum_k |G_{ki}|^2 \sum_k |G_{kj}|^2} \leqslant \|G\|^2$$

by Cauchy-Schwarz inequality. We further have

$$T_3^{(2)} \leqslant 2\sigma_{\max}^2 \|G\|^3 \sum_{i,j} |v_i \sigma_{ij}| \cdot |\sigma_{ij} G_{jw}| \leqslant 2\sigma_{\max}^2 \|G\|^3 \sqrt{\sum_{i,j} |v_i|^2 \sigma_{ij}^2} \sqrt{\sum_{i,j} \sigma_{ij}^2 |G_{jw}|^2}$$
$$\leqslant 2\sigma_{\max}^2 \|G\|^3 \sqrt{R_{\max}} \sqrt{R_{\max} \|Gw\|^2} \leqslant 2\sigma_{\max}^2 R_{\max} \|G\|^4 \leqslant \frac{8\sigma_{\max}^2}{9|z|^2}.$$

Combining the above estimates, we obtain

$$|\mathtt{A}| \leqslant \frac{76\sigma_{\max}^2}{9|z|^2} + \frac{76\sigma_{\max}^2}{9|z|^2} + \frac{8\sigma_{\max}^2}{9|z|^2} = \frac{160\sigma_{\max}^2}{9|z|^2} \leqslant \frac{18\sigma_{\max}^2}{|z|^2}.$$

The same estimate holds for $\bar{\mathtt{A}}$. Indeed, since $\partial_{ij}\overline{\mathcal{D}}_{vw} = \overline{\partial_{ij}\mathcal{D}_{vw}}$, the preceding argument applies with G, L, z replaced by $\overline{G}, \overline{L}, \overline{z}$. Hence,

$$|\bar{\mathtt{A}}| \leqslant \frac{18\sigma_{\max}^2}{|z|^2}.$$

Finally, we bound

$$|T^{(2)}| \leqslant (p-1)|\mathbb{E}\big[|\mathtt{A}| \cdot \mathcal{D}_{vw}^{p-2}\overline{\mathcal{D}}_{vw}^{p}\big]| + p|\mathbb{E}\big[|\bar{\mathtt{A}}| \cdot |\mathcal{D}_{vw}|^{2(p-1)}\big]|$$
$$\leqslant \frac{36p\sigma_{\max}^2}{|z|^2}\mathbb{E}[|\mathcal{D}_{vw}|^{2p-2}].$$

This completes the proof of Lemma C.3 (i).

Proof of Lemma C.3 (ii). For brevity, we suppress the harmless distinction between $\mathcal{D}_{vw}^{p-1}\overline{\mathcal{D}}_{vw}^{p}$ and $\mathcal{D}_{vw}^{2p-1}$. The argument with complex conjugates is identical, using $\partial_{ij}\overline{\mathcal{D}}_{vw} = \overline{\partial_{ij}\mathcal{D}_{vw}}$.

Recall that

$$X_2 = \sum_{i,j} v_i \, \frac{1}{2} \, \mathcal{C}_3(E_{ij}) \, \mathbb{E}\big[\partial_{ij}^2\big(G_{jw} \, \mathcal{D}_{vw}^{2p-1}\big)\big] \, .$$

Using the Leibniz rule and our assumption $|\mathcal{C}_3(E_{ij})| = |\mathbb{E}(E_{ij}^3)| \leqslant \beta K \sigma_{\max}\sigma_{ij}^2$, we get

$$|X_2| \leqslant \beta K \sigma_{\max} \sum_{s=0}^{2} \mathbb{E}\Big[\sum_{i,j} |v_i| \sigma_{ij}^2 |\partial_{ij}^{2-s}(G_{jw})| \cdot |\partial_{ij}^{s}(\mathcal{D}_{vw}^{2p-1})|\Big]. \tag{C.17}$$

We estimate the three values $s = 0, 1, 2$ separately.

• **Case 1:** $s = 0$**.** We estimate

$$\sum_{i,j} |v_i| \sigma_{ij}^2 |\partial_{ij}^2(G_{jw})|.$$

Using

$$\partial_{ij}^2 G = 2! G(\Delta^{ij} G)^2,$$

one obtains

$$\sum_{i,j} |v_i| \sigma_{ij}^2 |\partial_{ij}^2(G_{jw})| \leqslant 2 \sum_{i,j} |v_i| \sigma_{ij}^2 |e_j^{\mathrm{T}} G(\Delta^{ij} G)^2 w| = 2 \sum_j \big(\sum_i |v_i| \sigma_{ij}^2\big) \cdot |e_j^{\mathrm{T}} G(\Delta^{ij} G)^2 w|.$$

Let $\mathrm{v} = (|v_i|)$ and v is still a unit vector. By Cauchy-Schwartz, together with $\|\Delta^{ij}\| \leqslant 1$, we get

$$\begin{aligned}\sum_{i,j} |v_i| \sigma_{ij}^2 |\partial_{ij}^2(G_{jw})| &\leqslant 2 \sqrt{\sum_j \big(\sum_i \sigma_{ij}^2 \mathrm{v}_i\big)^2} \sqrt{\sum_j |e_j^{\mathrm{T}} G(\Delta^{ij} G)^2 w|^2} \\ &= 2\|\Sigma \mathrm{v}\| \cdot \|G(\Delta^{ij} G)^2 w\| \leqslant 2\|\Sigma\| \cdot \|G\|^3.\end{aligned}$$

Since $\|\Sigma\| \leqslant \sqrt{\|\Sigma\|_1 \cdot \|\Sigma\|_\infty} = R_{\max}$, $\|G\| \leqslant 2/|z|$ and $|z| \geqslant 6\sqrt{R_{\max}}$, this gives

$$\sum_{i,j} |v_i| \sigma_{ij}^2 |\partial_{ij}^2(G_{jw})| \leqslant 16 \frac{R_{\max}}{|z|^3} \leqslant \frac{4}{9} \frac{1}{|z|}.$$

Therefore, the contribution from the $s = 0$ term in (C.17) is bounded by

$$\frac{4}{9} \beta K \sigma_{\max} \frac{\mathbb{E}|\mathcal{D}_{vw}|^{2p-1}}{|z|}.$$

• **Case 2:** $s = 1$**.** We estimate

$$\mathbb{E}\Big[\sum_{i,j} |v_i| \sigma_{ij}^2 |\partial_{ij} G_{jw}| \cdot |\partial_{ij}(\mathcal{D}_{vw}^{2p-1})|\Big].$$

From (C.11), we first have

$$\partial_{ij} G_{jw} = (1 + \delta_{ij})^{-1} (G_{ij} G_{jw} + G_{jj} G_{iw}).$$

By the chain rule,

$$|\partial_{ij}(\mathcal{D}_{vw}^{2p-1})| \leqslant 2p |\mathcal{D}_{vw}|^{2p-2} |\partial_{ij} \mathcal{D}_{vw}|.$$

Recall from (C.16) that

$$\partial_{ij} \mathcal{D}_{vw} = (1 + \delta_{ij})^{-1} \Big(G_{\widetilde{v}i} G_{jw} + G_{\widetilde{v}j} G_{iw} - \mathcal{K}\Big),$$

where for brevity we define

$$\mathcal{K} := 2\sum_{s,k} v_s \sigma_{sk}^2 G_{ki} G_{kj} G_{sw}.$$

We also recall that $\widetilde{v}^{\mathrm{T}} := v^{\mathrm{T}}(zI - L)$ and $\|\widetilde{v}\| \leqslant \frac{19}{18}|z|$.

For vectors v, w, we will repeatedly use the following estimates:

$$\sum_i |G_{i\mathsf{w}}|^2 = \|G\mathsf{w}\|^2 \leqslant \|G\|^2 \|\mathsf{w}\|^2 \tag{C.18}$$

Consequently, by the Cauchy-Schwarz inequality,

$$\sum_i |\mathsf{v}_i G_{i\mathsf{w}}| \leqslant \|G\| \|\mathsf{v}\| \|\mathsf{w}\| \quad \text{and} \quad \sum_j |G_{\mathsf{v}j} G_{j\mathsf{w}}| \leqslant \|G\mathsf{v}\| \|G\mathsf{w}\| \leqslant \|G\|^2 \|\mathsf{v}\| \|\mathsf{w}\|. \tag{C.19}$$

It follows that

$$|\mathcal{K}| \leqslant 2\sigma_{\max}^2 \Big| \sum_s v_s G_{sw} \Big| \cdot \Big| \sum_k G_{ik} G_{kj} \Big| \leqslant 2\sigma_{\max}^2 \|G\|^3. \tag{C.20}$$

Now we are ready to estimate

$$\begin{aligned} \sum_{i,j} \big| v_i \sigma_{ij}^2 (\partial_{ij} G_{jw})(\partial_{ij} \mathcal{D}_{vw}) \big| \leqslant & \sum_{i,j} \big| v_i \sigma_{ij}^2 (G_{ij} G_{jw} + G_{jj} G_{iw})(G_{\widetilde{v}i} G_{jw} + G_{\widetilde{v}j} G_{iw}) \big| \\ & + \sum_{i,j} |v_i \sigma_{ij}^2 (G_{ij} G_{jw} + G_{jj} G_{iw}) \mathcal{K}|. \end{aligned} \tag{C.21}$$

For the first term on the right-hand side of (C.21), expanding the product yields four terms. We bound each respectively using (C.18) and (C.19):

$$\sum_{i,j} |v_i \sigma_{ij}^2| |G_{ij} G_{jw} G_{\widetilde{v}i} G_{jw}| \leqslant \sigma_{\max}^2 \|G\| \sum_i |v_i G_{\widetilde{v}i}| \sum_j |G_{jw}|^2 \leqslant \sigma_{\max}^2 \|G\|^4 \|\widetilde{v}\|.$$

$$\sum_{i,j} |v_i \sigma_{ij}^2| |G_{ij} G_{jw} G_{\widetilde{v}j} G_{iw}| \leqslant \sigma_{\max}^2 \|G\| \sum_i |v_i G_{iw}| \sum_j |G_{jw}| |G_{\widetilde{v}j}| \leqslant \sigma_{\max}^2 \|G\|^4 \|\widetilde{v}\|.$$

$$\sum_{i,j} |v_i \sigma_{ij}^2| |G_{jj} G_{iw} G_{\widetilde{v}i} G_{jw}| \leqslant \sigma_{\max} \|G\|^2 \sum_i |v_i G_{\widetilde{v}i}| \sum_j |\sigma_{ij} G_{jw}| \leqslant \sigma_{\max} \sqrt{R_{\max}}\, \|G\|^4 \|\widetilde{v}\|.$$

$$\sum_{i,j} |v_i \sigma_{ij}^2| |G_{jj} G_{iw} G_{\widetilde{v}j} G_{iw}| \leqslant \sigma_{\max} \|G\|^2 \sum_i |v_i G_{iw}| \sum_j |\sigma_{ij} G_{\widetilde{v}j}| \leqslant \sigma_{\max} \sqrt{R_{\max}}\, \|G\|^4 \|\widetilde{v}\|.$$

Summing these estimates and using the fact that $\sigma_{\max}^2 \leqslant \sigma_{\max}\sqrt{R_{\max}}$, we obtain

$$\sum_{i,j} \big| v_i \sigma_{ij}^2 (G_{ij} G_{jw} + G_{jj} G_{iw})(G_{\widetilde{v}i} G_{jw} + G_{\widetilde{v}j} G_{iw}) \big| \leqslant 4\sigma_{\max} \sqrt{R_{\max}}\, \|G\|^4 \|\widetilde{v}\|.$$

For the second term on the right-hand side of (C.21), note that by (C.20)

$$\sum_{i,j}|v_i\sigma_{ij}^2(G_{ij}G_{jw}+G_{jj}G_{iw})\mathcal{K}| \leqslant 2\sigma_{\max}^2\|G\|^3\sum_{i,j}|v_i\sigma_{ij}^2|(|G_{ij}G_{jw}|+|G_{jj}G_{iw}|)$$
$$\leqslant 4\sigma_{\max}^2R_{\max}\|G\|^5,$$

where in the last inequality, we apply (C.19) to bound

$$\sum_{i,j}|v_i\sigma_{ij}^2||G_{ij}||G_{jw}| \leqslant \sqrt{\sum_{i,j}v_i^2\sigma_{ij}^2|G_{ij}|^2}\sqrt{\sum_{i,j}\sigma_{ij}^2|G_{jw}|^2} \leqslant \sigma_{\max}\sqrt{R_{\max}}\|G\|^2$$

and

$$\sum_{i,j}|v_i\sigma_{ij}^2||G_{jj}G_{iw}| \leqslant \|G\|\sum_i|v_i||G_{iw}|\sum_j\sigma_{ij}^2 \leqslant R_{\max}\|G\|^2.$$

Combining the above estimates, we have

$$\sum_{i,j}|v_i\sigma_{ij}^2(\partial_{ij}G_{jw})(\partial_{ij}\mathcal{D}_{vw})| \leqslant 4\sigma_{\max}\sqrt{R_{\max}}\|G\|^4\|\widetilde{v}\| + 4\sigma_{\max}^2R_{\max}\|G\|^5.$$

Using (5.6) and our assumption $|z| \geqslant 6\sqrt{R_{\max}}$, we further have

$$\sum_{i,j}\big|v_i\sigma_{ij}^2(\partial_{ij}G_{jw})(\partial_{ij}\mathcal{D}_{vw})\big| \leqslant 4\sigma_{\max}\frac{|z|}{6}\frac{2^4}{|z|^4}\frac{19|z|}{18} + 4\sigma_{\max}\frac{|z|^2}{36}\frac{2^5}{|z|^5} \leqslant \frac{64\sigma_{\max}}{|z|^2}.$$

Consequently, the contribution from the $s=1$ term in (C.17) is bounded by

$$\beta K\sigma_{\max}\mathbb{E}\Big[\sum_{i,j}|v_i|\sigma_{ij}^2|\partial_{ij}G_{jw}|\cdot|\partial_{ij}(\mathcal{D}_{vw}^{2p-1})|\Big] \leqslant 128p\beta K\sigma_{\max}^2\frac{\mathbb{E}|\mathcal{D}_{vw}|^{2p-2}}{|z|^2}.$$

• **Case 3:** $s=2$**.** We estimate

$$\mathbb{E}\Big[\sum_{i,j}|v_i|\sigma_{ij}^2|G_{jw}||\partial_{ij}^2(\mathcal{D}_{vw}^{2p-1})|\Big].$$

The second derivative gives

$$|\partial_{ij}^2\left(\mathcal{D}_{vw}^{2p-1}\right)| \leqslant 2p|\mathcal{D}_{vw}|^{2p-2}|\partial_{ij}^2\mathcal{D}_{vw}| + 4p^2|\mathcal{D}_{vw}|^{2p-3}|\partial_{ij}\mathcal{D}_{vw}|^2.$$

Hence,

$$\mathbb{E}\Big[\sum_{i,j}|v_i|\sigma_{ij}^2|G_{jw}||\partial_{ij}^2(\mathcal{D}_{vw}^{2p-1})|\Big]$$
$$\leqslant 2p\mathbb{E}\Big[\sum_{i,j}|v_i|\sigma_{ij}^2|G_{jw}||\partial_{ij}^2\mathcal{D}_{vw}||\mathcal{D}_{vw}|^{2p-2}\Big] + 4p^2\mathbb{E}\Big[\sum_{i,j}|v_i|\sigma_{ij}^2|G_{jw}||\partial_{ij}\mathcal{D}_{vw}|^2|\mathcal{D}_{vw}|^{2p-3}\Big] \quad \text{(C.22)}$$

We first estimate

$$\sum_{i,j}|v_i|\sigma_{ij}^2|G_{jw}||\partial_{ij}^2\mathcal{D}_{vw}|.$$

From (C.16), we have

$$\partial_{ij}^2 \mathcal{D}_{vw} = (1+\delta_{ij})^{-1}\Big[\partial_{ij}(G_{\widetilde{v}i}G_{jw}) + \partial_{ij}(G_{\widetilde{v}j}G_{iw}) - \partial_{ij}\mathcal{K}\Big]$$

and thus

$$\begin{aligned}\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}(\partial_{ij}^2\mathcal{D}_{vw})| \leqslant \sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}\partial_{ij}(G_{\widetilde{v}i}G_{jw})| + \sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}\partial_{ij}(G_{\widetilde{v}j}G_{iw})| \\ + \sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}\partial_{ij}\mathcal{K}|. \qquad \text{(C.23)}\end{aligned}$$

The first two terms on the right-hand side of (C.23) can be estimated similarly. We provide the calculation for the first term. By (C.11),

$$\begin{aligned}&\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}\partial_{ij}(G_{\widetilde{v}i}G_{jw})| \\ &\leqslant \sum_{i,j}\Big[\big|v_i\sigma_{ij}^2 G_{jw}^2(G_{\widetilde{v}i}G_{ij} + G_{\widetilde{v}j}G_{ii})\big| + \big|v_i\sigma_{ij}^2 G_{jw}G_{\widetilde{v}i}(G_{ij}G_{jw} + G_{jj}G_{iw})\big|\Big].\end{aligned}$$

Note that each term on the right-hand side is bounded by $(\sigma_{\max}^2 + \sigma_{\max}\sqrt{R_{\max}})\|G\|^4\|\widetilde{v}\|$ by applying (C.18) and (C.19). For instance,

$$\begin{aligned}&\sum_{i,j}\big|v_i\sigma_{ij}^2 G_{jw}^2(G_{\widetilde{v}i}G_{ij} + G_{\widetilde{v}j}G_{ii})\big| \\ &\leqslant \sigma_{\max}^2\|G\|\sum_i|v_iG_{\widetilde{v}i}|\sum_j|G_{jw}|^2 + \sigma_{\max}\|G\|^2\|\widetilde{v}\|\sum_j|G_{jw}|^2\sum_i|v_i\sigma_{ij}| \\ &\leqslant (\sigma_{\max}^2 + \sigma_{\max}\sqrt{R_{\max}})\|G\|^4\|\widetilde{v}\|.\end{aligned}$$

Continuing from (C.23), we have

$$\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}(\partial_{ij}^2\mathcal{D}_{vw})| \leqslant 8\sigma_{\max}\sqrt{R_{\max}}\,\|G\|^4\|\widetilde{v}\| + \sum_{i,j}|v_iG_{jw}\partial_{ij}\mathcal{K}|.$$

It remains to estimate $\sum_{i,j}|v_iG_{jw}\partial_{ij}\mathcal{K}|$, where

$$\partial_{ij}\mathcal{K} = 2\sum_{s,k}v_s\sigma_{sk}^2\partial_{ij}(G_{ki}G_{kj}G_{sw}).$$

We first estimate $|\partial_{ij}\mathcal{K}|$. Expanding $\partial_{ij}(G_{ki}G_{kj}G_{sw})$ using the product rule and (C.11) yields several terms, each of which admits a similar treatment. We present the calculation for one representative term. For instance,

$$\begin{aligned}\sum_{s,k}|v_s\sigma_{sk}^2(\partial_{ij}G_{ki})G_{kj}G_{sw}| &\leqslant \sigma_{\max}^2\sum_{s,k}|v_s(G_{ki}G_{jj} + G_{kj}G_{ij})G_{kj}G_{sw}| \\ &\leqslant \sigma_{\max}^2\|G\|\sum_s|v_sG_{sw}|\sum_k(|G_{ki}| + |G_{kj}|)|G_{kj}| \\ &\leqslant 2\sigma_{\max}^2\|G\|^4\end{aligned}$$

where we applied (C.18) and (C.19) in the last inequality. Hence,

$$|\partial_{ij}\mathcal{K}| \leqslant 12\,\sigma_{\max}^2\|G\|^4$$

and by Cauchy-Schwarz and (C.18),

$$\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}\partial_{ij}\mathcal{K}| \leqslant 12\sigma_{\max}^2\|G\|^4\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}| \leqslant 12\sigma_{\max}^2\|G\|^4\cdot R_{\max}\|G\| = 12\sigma_{\max}^2 R_{\max}\|G\|^5.$$

Finally, using $|z| \geqslant 6\sqrt{R_{\max}}$ and $\|G\| \leqslant 2/|z|$, we have

$$\begin{aligned}\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}(\partial_{ij}^2\mathcal{D}_{vw})| &\leqslant 8\sigma_{\max}\sqrt{R_{\max}}\|G\|^4\|\widetilde{v}\| + 12\sigma_{\max}^2 R_{\max}\|G\|^5\\ &\leqslant 8\sigma_{\max}\frac{|z|}{6}\frac{2^4}{|z|^4}\frac{19|z|}{18} + 12\sigma_{\max}\frac{|z|^3}{6^3}\frac{2^5}{|z|^5} < 25\frac{\sigma_{\max}}{|z|^2}. \qquad \text{(C.24)}\end{aligned}$$

Next, we bound

$$\sum_{i,j}|v_i|\sigma_{ij}^2|G_{jw}||\partial_{ij}\mathcal{D}_{vw}|^2.$$

Continuing from (C.16), we get

$$(\partial_{ij}\mathcal{D}_{vw})^2 = (1+\delta_{ij})^{-2}\Big[(G_{\widetilde{v}i}G_{jw} + G_{\widetilde{v}j}G_{iw})^2 + \mathcal{K}^2 - 2\mathcal{K}(G_{\widetilde{v}i}G_{jw} + G_{\widetilde{v}j}G_{iw})\Big],$$

where

$$\mathcal{K} = 2\sum_{s,k}v_s\sigma_{sk}^2 G_{ki}G_{kj}G_{sw}.$$

Plugging this expansion into $\sum_{i,j}|v_iG_{jw}(\partial_{ij}\mathcal{D}_{vw})^2|$, we first observe from (C.20) that

$$\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}||\mathcal{K}|^2 \leqslant (2\sigma_{\max}^2\|G\|^3)^2\sum_{i,j}|v_i\sigma_{ij}^2||G_{jw}| \leqslant 4\sigma_{\max}^4\|G\|^6\cdot R_{\max}\|G\|.$$

Next, using (C.18) and (C.19), we also have

$$\begin{aligned}&2\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}(G_{\widetilde{v}i}G_{jw} + G_{\widetilde{v}j}G_{iw})||\mathcal{K}|\\ &\leqslant 4\sigma_{\max}^4\|G\|^3\left(\sum_{i,j}|v_iG_{\widetilde{v}i}|\cdot|G_{jw}|^2 + \sum_i|v_iG_{iw}|\sum_j|G_{jw}G_{\widetilde{v}j}|\right)\\ &\leqslant 8\sigma_{\max}^4\|G\|^6\|\widetilde{v}\|.\end{aligned}$$

Finally, we claim that

$$\sum_{i,j}|v_i\sigma_{ij}^2 G_{jw}(G_{\widetilde{v}i}G_{jw} + G_{\widetilde{v}j}G_{iw})^2| \leqslant 4\sigma_{\max}^2\|G\|^5\|\widetilde{v}\|^2.$$

To see this, we expand the square in the summation and show that each term is bounded by $\|G\|^5\|\widetilde{v}\|^2$, using a similar calculation with (C.18) and (C.19). For instance,

$$\sum_{i,j}|v_iG_{jw}G_{\widetilde{v}i}^2G_{jw}^2| \leqslant \|G\|^2\|\widetilde{v}\|\sum_i|v_iG_{\widetilde{v}i}|\sum_j|G_{jw}|^2 \leqslant \|G\|^5\|\widetilde{v}\|^2,$$

and the remaining terms are estimated in the same manner.

Hence, together with $\|\widetilde{v}\| \leqslant \frac{19}{18}|z|$, we get

$$\sum_{i,j} |v_i \sigma_{ij}^2 G_{jw} (\partial_{ij} \mathcal{D}_{vw})^2| \leqslant 4\sigma_{\max}^4 R_{\max} \|G\|^7 + 8\sigma_{\max}^4 \|G\|^6 \|\widetilde{v}\| + 4\sigma_{\max}^2 \|G\|^5 \|\widetilde{v}\|^2$$
$$\leqslant \frac{2^3 \sigma_{\max}^2}{|z|^3} \Big(4\frac{|z|^4}{6^4}\frac{2^4}{|z|^4} + 8\frac{|z|^2}{6^2}\frac{2^3}{|z|^3}\frac{19|z|}{18} + 4\frac{2^2}{|z|^2}\frac{19^2|z|^2}{18^2}\Big) < 175\frac{\sigma_{\max}^2}{|z|^3}. \tag{C.25}$$

We continue from (C.22) to get

$$\mathbb{E}\Big[\sum_{i,j} |v_i| \sigma_{ij}^2 |G_{jw}| |\partial_{ij}^2 (\mathcal{D}_{vw}^{2p-1})|\Big] \leqslant 50p\sigma_{\max} \frac{\mathbb{E}|\mathcal{D}_{vw}|^{2p-2}}{|z|^2} + 700p^2 \sigma_{\max}^2 \frac{\mathbb{E}|\mathcal{D}_{vw}|^{2p-3}}{|z|^3}.$$

Thus, the contribution from the $s = 2$ term in (C.17) is bounded by

$$\beta K \sigma_{\max} \mathbb{E}\Big[\sum_{i,j} |v_i| \sigma_{ij}^2 |G_{jw}| |\partial_{ij}^2 (\mathcal{D}_{vw}^{2p-1})|\Big]$$
$$\leqslant 50p\beta K \sigma_{\max}^2 \frac{\mathbb{E}|D_{vw}|^{2p-2}}{|z|^2} + 700p^2 \beta K \sigma_{\max}^3 \frac{\mathbb{E}|\mathcal{D}_{vw}|^{2p-3}}{|z|^3}.$$

Combining all the above estimates completes the proof of Lemma C.3 (ii).

A third-order bound for the remainder term. Before turning to the remainder term, we record an additional derivative estimate required by the cumulant remainder formula in Lemma C.1. This estimate serves as a third-order analogue of those used in bounding X_2 above, corresponding to the third-order derivatives of

$$G_{jw} \mathcal{D}_{vw}^{2p-1}.$$

The argument follows the structure of the $s = 0, 1, 2$ cases in the proof of Lemma C.3 (ii). We isolate this estimate here for use in the remainder analysis.

Recall from the truncation parameter $\mathtt{L}_n$ from (C.3). For $1 \leqslant i, j \leqslant n$ and $|x| \leqslant \mathtt{L}_n$, let

$$E^{ij(x)} := E + (x - E_{ij})\Delta^{ij}, \qquad G^{ij(x)}(z) := (zI - E^{ij(x)})^{-1}.$$

Define

$$L^{ij(x)} := \operatorname{diag}(l_1^{ij(x)}, \ldots, l_n^{ij(x)}) \qquad \text{with} \quad l_s^{ij(x)} := \sum_{k=1}^n \sigma_{sk}^2 G_{kk}^{ij(x)},$$

and

$$\mathcal{D}^{ij(x)} := (E^{ij(x)} - L^{ij(x)}) G^{ij(x)}(z).$$

We use the notation $\mathcal{D}_{vw}^{ij(x)} = v^{\mathrm{T}} \mathcal{D}^{ij(x)} w$ and $G_{vw}^{ij(x)} = v^{\mathrm{T}} G^{ij(x)} w$. Set

$$f_{ij}(E^{ij(x)}) := G_{jw}^{ij(x)} \left(\mathcal{D}_{vw}^{ij(x)}\right)^{2p-1}.$$

Recall from (C.4) the event

$$\Omega_n := \left\{ \|E\| \leqslant 3.5\sqrt{R_{\max}}, \quad \max_{a,b} |E_{ab}| \leqslant 2\mathtt{L}_n \right\}.$$

The goal is to prove the following estimate on the event Ω_n (the support of $\chi(E)$):

$$\sum_{i,j} |v_i| \sigma_{ij}^2 \sup_{|x| \leqslant \mathtt{L}_n} \left| \partial_{ij}^3 f_{ij}(E^{ij(x)}) \right| \leqslant C \sum_{m=1}^{4} p^{m-1} \frac{\sigma_{\max}^{m-1}}{|z|^{m+1}} \left(|\mathcal{D}_{vw}| + 22 \frac{\mathtt{L}_n}{|z|} \right)^{2p-m} \tag{C.26}$$

for an absolute constant $C > 0$.

We start with some preliminary estimates. On the event Ω_n, we have, for $|x| \leqslant \mathtt{L}_n$,

$$\|E^{ij(x)}\| \leqslant \|E\| + (|x| + |E_{ij}|)\|\Delta_{ij}\| \leqslant 3.5\sqrt{R_{\max}} + 3\mathtt{L}_n \leqslant 4\sqrt{R_{\max}} \tag{C.27}$$

provided the constant C_0 in Assumption 1.2 is chosen sufficiently large. Hence, for $|z| \geqslant 6\sqrt{R_{\max}}$, by the Neumann expansion, we have

$$\|G^{ij(x)}(z)\| \leqslant \frac{3}{|z|}. \tag{C.28}$$

Next, we show that

$$\sup_{|x| \leqslant \mathtt{L}_n} |\mathcal{D}_{vw}^{ij(x)} - \mathcal{D}_{vw}| \leqslant 22 \frac{\mathtt{L}_n}{|z|}.$$

To see this, for $|x| \leqslant \mathtt{L}_n$, we bound

$$|\mathcal{D}_{vw}^{ij(x)} - \mathcal{D}_{vw}| = |v^{\mathrm{T}}(\mathcal{D}^{ij(x)} - \mathcal{D})w| \leqslant \|\mathcal{D}^{ij(x)} - \mathcal{D}\|.$$

From the definitions of $\mathcal{D}^{ij(x)}$ and $\mathcal{D}$, we further have

$$\begin{aligned} |\mathcal{D}_{vw}^{ij(x)} - \mathcal{D}_{vw}| &\leqslant \|(E^{ij(x)} - L^{ij(x)})G^{ij(x)}(z) - (E - L)G(z)\| \\ &= \|[(E^{ij(x)} - E) - (L^{ij(x)} - L)]G^{ij(x)}(z) + (E - L)(G^{ij(x)}(z) - G(z))\| \\ &\leqslant (\|E^{ij(x)} - E\| + \|L^{ij(x)} - L\|)\|G^{ij(x)}(z)\| + (\|E\| + \|L\|)\|G^{ij(x)}(z) - G(z)\|. \end{aligned}$$

For simplicity, denote $P^{ij} := E^{ij(x)} - E = (x - E_{ij})\Delta^{ij}$. Similar to (C.27), we have

$$\|E^{ij(x)} - E\| = \|P^{ij}\| \leqslant (|x| + |E_{ij}|)\|\Delta_{ij}\| \leqslant 3\mathtt{L}_n.$$

Next, by the resolvent identity, we have

$$G^{ij(x)}(z) - G(z) = G^{ij(x)}(z)P^{ij}G(z)$$

Thus, by (C.28) and (5.6),

$$\|G^{ij(x)}(z) - G(z)\| = \|G^{ij(x)}(z)\| \cdot \|P^{ij}\| \cdot \|G(z)\| \leqslant 18 \frac{\mathtt{L}_n}{|z|^2}.$$

Then

$$\|L^{ij(x)} - L\| = \max_s |l_s^{ij(x)} - l_s| = \max_s \sum_{k=1}^{n} \sigma_{sk}^2 |G_{kk}^{ij(x)}(z) - G_{kk}(z)|$$
$$\leqslant R_{\max}\|G^{ij(x)}(z) - G(z)\| \leqslant 18\mathtt{L}_n \frac{R_{\max}}{|z|^2} \leqslant \frac{1}{2}\mathtt{L}_n$$

by the assumption $|z| \geqslant 6\sqrt{R_{\max}}$. Similarly,

$$\|L\| = \max_s |l_s| \leqslant \max_s \sum_{k=1}^{n} \sigma_{sk}^2 |G_{kk}(z)| \leqslant R_{\max}\|G(z)\| \leqslant 2\frac{R_{\max}}{|z|}.$$

It follows that

$$\|E\| + \|L\| \leqslant 3.5\sqrt{R_{\max}} + 2\frac{R_{\max}}{|z|} \leqslant \left(\frac{7}{12} + \frac{1}{18}\right)|z|.$$

Combining the above estimates, we conclude that

$$|\mathcal{D}_{vw}^{ij(x)} - \mathcal{D}_{vw}| \leqslant 10.5\frac{\mathtt{L}_n}{|z|} + 18\left(\frac{7}{12} + \frac{1}{18}\right)\frac{\mathtt{L}_n}{|z|} = 22\frac{\mathtt{L}_n}{|z|}.$$

Thus

$$\sup_{|x| \leqslant \mathtt{L}_n} |\mathcal{D}_{vw}^{ij}(x)| \leqslant |\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_n}{|z|}. \tag{C.29}$$

Now, we proceed to the proof of (C.26). By the Leibniz rule,

$$\partial_{ij}^3 f_{ij}(E^{ij(x)}) = \sum_{s=0}^{3} \binom{3}{s} \left(\partial_{ij}^{3-s} G_{jw}^{ij(x)}\right) \left(\partial_{ij}^{s} (\mathcal{D}_{vw}^{ij(x)})^{2p-1}\right).$$

We estimate the four cases $s = 0, 1, 2, 3$ separately. The cases $s = 0, 1, 2$ follow similar arguments to those used for X_2 in (C.17).

First, for $s = 0$, using

$$\partial_{ij}^3 G^{ij(x)} = 3! G^{ij(x)} (\Delta^{ij} G^{ij(x)})^3$$

and the same estimate used for the $s = 0$ term in the X_2 bound, we get

$$\sum_{i,j} |v_i| \sigma_{ij}^2 |\partial_{ij}^3 G_{jw}^{ij(x)}| \leqslant 6R_{\max}\|G^{ij(x)}\|^4 \leqslant 6R_{\max}\frac{3^4}{|z|^4} \leqslant \frac{27}{2|z|^2}.$$

Therefore,

$$\sum_{i,j} |v_i| \sigma_{ij}^2 |\partial_{ij}^3 G_{jw}^{ij(x)}| \cdot |(\mathcal{D}_{vw}^{ij(x)})^{2p-1}| \leqslant \frac{27}{2|z|^2} |\mathcal{D}_{vw}^{ij(x)}|^{2p-1}.$$

Next, for $s = 1$, since

$$|\partial_{ij}(\mathcal{D}_{vw}^{ij(x)})^{2p-1}| \leqslant 2p|\mathcal{D}_{vw}^{ij(x)}|^{2p-2} |\partial_{ij}\mathcal{D}_{vw}^{ij(x)}|,$$

we aim to bound

$$\sum_{i,j} |v_i|\sigma_{ij}^2|\partial_{ij}^2 G_{jw}^{ij}(x)||\partial_{ij}\mathcal{D}_{vw}^{ij(x)}|.$$

This is proved exactly as the $s = 1$ estimate for X_2, but with an extra factor of G. We sketch the key steps here. Expand

$$\partial_{ij}\mathcal{D}_{vw}^{ij(x)} = (1+\delta_{ij})^{-1}\left(G_{\widetilde{v}_x i}^{ij(x)}G_{jw}^{ij(x)} + G_{\widetilde{v}_x j}^{ij(x)}G_{iw}^{ij(x)} - \mathcal{K}_x\right),$$

where $\widetilde{v}_x^{\mathrm{T}} := v^{\mathrm{T}}(zI - L^{ij(x)})$ and

$$\mathcal{K}_x := 2\sum_{s,k} v_s\sigma_{sk}^2 G_{ki}^{ij(x)}G_{kj}^{ij(x)}G_{sw}^{ij(x)}.$$

Since

$$\|L^{ij(x)}\| \leqslant \max_s \sum_{k=1}^n \sigma_{sk}^2|G^{ij(x)}(z)| \leqslant R_{\max}\|G^{ij(x)}(z)\| \leqslant 3\frac{R_{\max}}{|z|} \leqslant \frac{1}{12}|z|,$$

we have $\|\widetilde{v}_x^{\mathrm{T}}\| \leqslant \frac{13}{12}|z|$. Similar to (C.20), we obtain

$$|\mathcal{K}_x| \leqslant 2\sigma_{\max}^2\|G^{ij(x)}\|^3.$$

For the second derivative of the resolvent

$$\partial_{ij}^2 G_{jw}^{ij(x)} = 2(1+\delta_{ij})^{-2}\Big[\big((G_{ij}^{ij(x)})^2 + G_{ii}^{ij(x)}G_{jj}^{ij(x)}\big)G_{jw}^{ij(x)} + 2G_{ij}^{ij(x)}G_{jj}^{ij(x)}G_{iw}^{ij(x)}\Big],$$

we apply the crude upper bound

$$|\partial_{ij}^2 G_{jw}^{ij(x)}| \leqslant 4\|G^{ij(x)}\|^2\big(|G_{jw}^{ij(x)}| + |G_{iw}^{ij(x)}|\big).$$

Multiplying this by the expansion of $\partial_{ij}\mathcal{D}_{vw}^{ij(x)}$ and applying the identical Cauchy-Schwarz estimates used for X_2, we obtain

$$\begin{aligned}\sum_{i,j} |v_i|\sigma_{ij}^2|\partial_{ij}^2 G_{jw}^{ij(x)}||\partial_{ij}\mathcal{D}_{vw}^{ij(x)}| &\leqslant 16\sigma_{\max}\sqrt{R_{\max}}\|G^{ij(x)}\|^5\|\widetilde{v}_x\| + 16\sigma_{\max}^2 R_{\max}\|G^{ij(x)}\|^6 \\ &\leqslant 16\sigma_{\max}\left(\frac{|z|}{6}\right)\frac{3^5}{|z|^5}\left(\frac{13|z|}{12}\right) + 16\sigma_{\max}\left(\frac{|z|^3}{216}\right)\frac{3^6}{|z|^6} \\ &\leqslant 702\frac{\sigma_{\max}}{|z|^3} + 54\frac{\sigma_{\max}}{|z|^3} = 756\frac{\sigma_{\max}}{|z|^3}.\end{aligned}$$

Hence the $s = 1$ contribution is bounded by

$$1512p\frac{\sigma_{\max}}{|z|^3}|\mathcal{D}_{vw}^{ij(x)}|^{2p-2}.$$

For $s = 2$, we use

$$|\partial_{ij}^2\left((\mathcal{D}_{vw}^{ij(x)})^{2p-1}\right)| \leqslant 2p|\mathcal{D}_{vw}^{ij(x)}|^{2p-2}|\partial_{ij}^2\mathcal{D}_{vw}^{ij(x)}| + 4p^2|\mathcal{D}_{vw}^{ij(x)}|^{2p-3}|\partial_{ij}\mathcal{D}_{vw}^{ij(x)}|^2.$$

The first estimate needed is

$$\sum_{i,j} |v_i|\sigma_{ij}^2|\partial_{ij}G_{jw}^{ij}(x)||\partial_{ij}^2\mathcal{D}_{vw}^{ij}(x)|.$$

Using $|\partial_{ij}G_{jw}^{ij(x)}| \leqslant 2\|G^{ij(x)}\|(|G_{jw}^{ij(x)}| + |G_{iw}^{ij(x)}|)$ and multiplying by the bounds established in (C.24), we pull out an extra factor of $2\|G^{ij(x)}\|$ and double the sum to account for $|G_{iw}|$:

$$\begin{aligned}\sum_{i,j} |v_i|\sigma_{ij}^2|\partial_{ij}G_{jw}^{ij(x)}||\partial_{ij}^2\mathcal{D}_{vw}^{ij(x)}| &\leqslant 32\sigma_{\max}\sqrt{R_{\max}}\|G^{ij(x)}\|^5\|\widetilde{v}_x\| + 48\sigma_{\max}^2 R_{\max}\|G^{ij(x)}\|^6 \\ &\leqslant 1404\frac{\sigma_{\max}}{|z|^3} + 162\frac{\sigma_{\max}}{|z|^3} = 1566\frac{\sigma_{\max}}{|z|^3}.\end{aligned}$$

We also require the estimate for

$$\sum_{i,j} |v_i|\sigma_{ij}^2|\partial_{ij}G_{jw}^{ij}(x)||\partial_{ij}\mathcal{D}_{vw}^{ij(x)}|^2.$$

Again, using $|\partial_{ij}G_{jw}^{ij(x)}| \leqslant 2\|G^{ij(x)}\|(|G_{jw}^{ij(x)}| + |G_{iw}^{ij(x)}|)$ to the estimates from (C.25) yields:

$$\begin{aligned}\sum_{i,j} |v_i|\sigma_{ij}^2|\partial_{ij}G_{jw}^{ij(x)}||\partial_{ij}\mathcal{D}_{vw}^{ij(x)}|^2 &\leqslant 24\sigma_{\max}^2\|G^{ij(x)}\|^6\|\widetilde{v}_x\|^2 + 16\sigma_{\max}^4 R_{\max}\|G^{ij(x)}\|^8 \\ &\leqslant 24\sigma_{\max}^2\frac{3^6}{|z|^6}\left(\frac{169|z|^2}{144}\right) + 16\sigma_{\max}^2\left(\frac{|z|^4}{1296}\right)\frac{3^8}{|z|^8} \\ &\leqslant C\frac{\sigma_{\max}^2}{|z|^4}.\end{aligned}$$

Therefore, the total $s = 2$ contribution is bounded by

$$Cp\frac{\sigma_{\max}}{|z|^3}|\mathcal{D}_{vw}^{ij(x)}|^{2p-2} + Cp^2\frac{\sigma_{\max}^2}{|z|^4}|\mathcal{D}_{vw}^{ij(x)}|^{2p-3}.$$

Finally, for $s = 3$, applying the generalized chain rule (Fa di Bruno's formula [45, p. 137]) to the third derivative of $(\mathcal{D}_{vw}^{ij(x)})^{2p-1}$ yields:

$$\begin{aligned}\left|\partial_{ij}^3\left((\mathcal{D}_{vw}^{ij(x)})^{2p-1}\right)\right| \leqslant\ & 2p|\mathcal{D}_{vw}^{ij(x)}|^{2p-2}|\partial_{ij}^3\mathcal{D}_{vw}^{ij(x)}| \\ &+ 12p^2|\mathcal{D}_{vw}^{ij(x)}|^{2p-3}|\partial_{ij}\mathcal{D}_{vw}^{ij(x)}||\partial_{ij}^2\mathcal{D}_{vw}^{ij(x)}| \\ &+ 8p^3|\mathcal{D}_{vw}^{ij(x)}|^{2p-4}|\partial_{ij}\mathcal{D}_{vw}^{ij(x)}|^3.\end{aligned}$$

Bounding this requires three separate estimates:

$$\sum_{i,j} |v_i|\sigma_{ij}^2|G_{jw}^{ij(x)}||\partial_{ij}^3\mathcal{D}_{vw}^{ij(x)}| \leqslant C_1\frac{\sigma_{\max}}{|z|^3}, \tag{C.30}$$

$$\sum_{i,j} |v_i|\sigma_{ij}^2|G_{jw}^{ij(x)}||\partial_{ij}\mathcal{D}_{vw}^{ij(x)}||\partial_{ij}^2\mathcal{D}_{vw}^{ij(x)}| \leqslant C_2\frac{\sigma_{\max}^2}{|z|^4}, \tag{C.31}$$

$$\sum_{i,j} |v_i|\sigma_{ij}^2|G_{jw}^{ij(x)}||\partial_{ij}\mathcal{D}_{vw}^{ij(x)}|^3 \leqslant C_3\frac{\sigma_{\max}^3}{|z|^5}, \tag{C.32}$$

where $C_1, C_2, C_3 > 0$ are absolute constants (which are all bounded 10^5).

We briefly sketch the proof of these estimates. For (C.30), from the identity $\mathcal{D} = (zI - L)G - I$, we have

$$\partial_{ij}^3 \mathcal{D} = (zI - L)\partial_{ij}^3 G - 3(\partial_{ij} L)\partial_{ij}^2 G - 3(\partial_{ij}^2 L)\partial_{ij} G - (\partial_{ij}^3 L)G.$$

Then we multiply this expansion with $|v_i|\sigma_{ij}^2 |G_{jw}^{ij(x)}|$ and sum over i, j. The first term is bounded by expanding $\partial_{ij}^3 G = 3! G(\Delta^{ij} G)^3$. The factor $|G_{jw}^{ij(x)}|$ combined with the variance weight generates one factor of $\sigma_{\max}$, while the remaining Green functions are controlled by (C.18) and (C.19). This yields the $C_1 \sigma_{\max} |z|^{-3}$ bound. The terms containing $\partial_{ij}^a L$ are handled by expanding

$$\partial_{ij}^a l_s = \sum_k \sigma_{sk}^2 \partial_{ij}^a G_{kk}^{ij(x)},$$

together with

$$|\partial_{ij}^a G_{kk}^{ij(x)}| \leqslant \widetilde{C}_a \|G^{ij(x)}\|^{a-1} \sum_{s \in \{i,j\}} |G_{ks}^{ij(x)}|^2.$$

and the Cauchy-Schwarz inequality. This proves (C.30).

The estimates (C.31) and (C.32) follow by combining the already proved bounds for $\partial_{ij} \mathcal{D}_{vw}^{ij(x)}$ and $\partial_{ij}^2 \mathcal{D}_{vw}^{ij(x)}$, and the factor $|G_{jw}^{ij(x)}|$. Note that each additional $\partial_{ij}\mathcal{D}$ contributes one factor $\sigma_{\max}/|z|$ in the estimate, exactly as in the $s = 2$ case of the X_2 estimate.

Consequently, the $s = 3$ contribution is bounded by

$$2p\, C_1 \frac{\sigma_{\max}}{|z|^3} |\mathcal{D}_{vw}^{ij(x)}|^{2p-2} + 12p^2\, C_2 \frac{\sigma_{\max}^2}{|z|^4} |\mathcal{D}_{vw}^{ij(x)}|^{2p-3} + 8p^3\, C_3 \frac{\sigma_{\max}^3}{|z|^5} |\mathcal{D}_{vw}^{ij(x)}|^{2p-4}.$$

Combining the four cases $s = 0, 1, 2, 3$ yields

$$\sum_{i,j} |v_i| \sigma_{ij}^2 \sup_{|x| \leqslant \mathtt{L}_n} \left| \partial_{ij}^3 \left(G_{jw}^{ij(x)} (\mathcal{D}_{vw}^{ij(x)})^{2p-1} \right) \right| \leqslant C \sum_{m=1}^{4} p^{m-1} \frac{\sigma_{\max}^{m-1}}{|z|^{m+1}} |\mathcal{D}_{vw}^{ij(x)}|^{2p-m}. \quad \text{(C.33)}$$

Using $|\mathcal{D}_{vw}^{ij(x)}| \leqslant |\mathcal{D}_{vw}| + 22\mathtt{L}_n/|z|$ from (C.29), we finally obtain

$$\sum_{i,j} |v_i| \sigma_{ij}^2 \sup_{|x| \leqslant \mathtt{L}_n} \left| \partial_{ij}^3 \left(G_{jw}^{ij(x)} (\mathcal{D}_{vw}^{ij(x)})^{2p-1} \right) \right| \leqslant C \sum_{m=1}^{4} p^{m-1} \frac{\sigma_{\max}^{m-1}}{|z|^{m+1}} \left(|\mathcal{D}_{vw}| + 22 \frac{\mathtt{L}_n}{|z|} \right)^{2p-m}.$$

This proves (C.26).

Proof of Lemma C.3 (iii). We estimate the remainder term in the cumulant expansion with $l = 2$. Recall from (C.13) that

$$\mathtt{R}_3 = \sum_{i,j} v_i \mathtt{R}_3^{ij}.$$

By Lemma C.1,

$$
\begin{aligned}
|\mathtt{R}_3| &\leqslant \sum_{i,j} |v_i \mathtt{R}_3^{ij}| \\
&\leqslant 16 \sum_{i,j} |v_i| \mathbb{E}\left[|E_{ij}|^4\right] \mathbb{E}\left[\sup_{|x|\leqslant t} |\partial_{ij}^3 f_{ij}(E^{ij} + x\Delta^{ij})|\right] && \text{(Term 1)} \\
&+ 16 \sum_{i,j} |v_i| \left(\mathbb{E}\left[\sup_{|x|\leqslant |E_{ij}|} |\partial_{ij}^3 f_{ij}(E^{ij(x)})|^2\right]\right)^{1/2} \left(\mathbb{E}\left[|E_{ij}|^8 \mathbf{1}_{|E_{ij}|>t}\right]\right)^{1/2} && \text{(Term 2)}
\end{aligned}
$$

We choose $t = \mathtt{L}_n$. In the general sub-Gaussian regime, $\mathtt{L}_n = 5\sqrt{D+6}K\sigma_{\max}\log n$, , while in the bounded sparse-entry regime, it is equal to zero. Here, as before, the cutoff χ is suppressed from the notation. Also, as explained in the beginning of Section C.1, terms with derivatives applied to χ are negligible.

Let us consider the general sub-Gaussian regime. We first estimate Term 1. Set

$$\theta := \beta K \sigma_{\max}.$$

By Assumption 1.1,

$$\mathbb{E}|E_{ij}|^4 \leqslant \beta^2 K^2 \sigma_{\max}^2 \sigma_{ij}^2 = \theta^2 \sigma_{ij}^2.$$

Hence, by (C.26), the contribution from Term 1 is bounded by

$$
\begin{aligned}
16 \sum_{i,j} &|v_i| \mathbb{E}\left[|E_{ij}|^4\right] \mathbb{E}\left[\sup_{|x|\leqslant \mathtt{L}_n} |\partial_{ij}^3 f_{ij}(E^{ij} + x\Delta^{ij})|\right] \\
&\leqslant 16\theta^2 \sum_{i,j} |v_i| \sigma_{ij}^2 \mathbb{E}\left[\sup_{|x|\leqslant \mathtt{L}_n} \left|\partial_{ij}^3 f_{ij}(E^{ij(x)})\right|\right] \\
&\leqslant C\theta^2 \sum_{m=1}^{4} p^{m-1} \frac{\sigma_{\max}^{m-1}}{|z|^{m+1}} \mathbb{E}\left[\left(|\mathcal{D}_{vw}| + 22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m}\right].
\end{aligned}
$$

It remains to estimate Term 2. By the sub-Gaussian tail assumption, we have

$$\left(\mathbb{E}\left[|E_{ij}|^8 \mathbf{1}_{\{|E_{ij}|>\mathtt{L}_n\}}\right]\right)^{1/2} \leqslant \theta^2 \sigma_{ij}^2 \exp(-10(D+6)\log^2 n). \tag{C.34}$$

To see this, note that the case $\sigma_{ij} = 0$ is trivial. For $\sigma_{ij} > 0$, write $E_{ij} = \sigma_{ij}\xi_{ij}$ with $\|\xi_{ij}\|_{\psi_2} \leqslant K$. A standard sub-Gaussian tail integration yields

$$
\begin{aligned}
\mathbb{E}\left[|E_{ij}|^8 \mathbf{1}_{\{|E_{ij}|>\mathtt{L}_n\}}\right] &\leqslant C(K\sigma_{ij})^8 \left(1 + \frac{\mathtt{L}_n}{K\sigma_{ij}}\right)^8 \exp\left(-\frac{\mathtt{L}_n^2}{K^2\sigma_{ij}^2}\right) \\
&= C\left(K\sigma_{ij} + \mathtt{L}_n\right)^8 \exp\left(-\frac{\mathtt{L}_n^2}{K^2\sigma_{ij}^2}\right).
\end{aligned} \tag{C.35}
$$

Because $\sigma_{ij} \leqslant \sigma_{\max}$ and $\mathtt{L}_n = 5\sqrt{D+6}K\sigma_{\max}\log n$, we bound

$$(K\sigma_{ij} + \mathtt{L}_n)^8 \leqslant CD^4 K^8 \sigma_{\max}^8 (\log n)^8.$$

Moreover, we write

$$\exp\left(-\frac{\mathrm{L}_n^2}{K^2\sigma_{ij}^2}\right) = \frac{\sigma_{ij}^4}{\sigma_{\max}^4}\left[\frac{\sigma_{\max}^4}{\sigma_{ij}^4}\exp\left(-25(D+6)\frac{\sigma_{\max}^2}{\sigma_{ij}^2}\log^2 n\right)\right]$$
$$\leqslant \frac{\sigma_{ij}^4}{\sigma_{\max}^4}\exp\big(-25(D+6)\log^2 n\big),$$

where we used the fact that the function $y \mapsto y^2\exp(-cy)$ is strictly decreasing for $y \geqslant 1$ and large $c > 0$, and then $y^2\exp(-cy) \leqslant \exp(-c)$.

Thus, the right-hand side of (C.35) is bounded by $CD^4K^8\sigma_{\max}^4\sigma_{ij}^4(\log n)^8\exp(-25(D+6)\log^2 n)$. To get the desired bound (C.34), we only need this quantity

$$D^4K^8\sigma_{\max}^4\sigma_{ij}^4(\log n)^8\exp(-25(D+6)\log^2 n) \leqslant \theta^4\sigma_{ij}^4\exp(-20(D+6)\log^2 n),$$

where $\theta = \beta K\sigma_{\max}$. Since $\beta \geqslant 1$, we require

$$\exp(25(D+6)\log^2 n) \geqslant D^4K^4\exp(20(D+6)\log^2 n). \tag{C.36}$$

We now use Assumption 1.2. Since $R_{\max} \leqslant n\sigma_{\max}^2$ and $\sqrt{R_{\max}} \geqslant C_0(D+8)K\sigma_{\max}\log n$, we have

$$K \leqslant \frac{\sqrt{n}}{C_0(D+8)\log n}.$$

Hence, (C.36) holds and (C.34) is proved.

Now, Term 2 is bounded by

$$16\sum_{i,j}|v_i|\left(\mathbb{E}\left[\sup_{|x|\leqslant|E_{ij}|}|\partial_{ij}^3 f_{ij}(E^{ij(x)})|^2\right]\right)^{1/2}\left(\mathbb{E}\left[|E_{ij}|^8\mathbf{1}_{|E_{ij}|>t}\right]\right)^{1/2}$$
$$\leqslant C\exp(-10(D+6)\log^2 n)\theta^2\sum_{i,j}\left(\mathbb{E}\left[\sup_{|x|\leqslant|E_{ij}|}\Big||v_i|\sigma_{ij}^2\partial_{ij}^3 f_{ij}(E^{ij(x)})\Big|^2\right]\right)^{1/2}.$$

Recall that the cutoff χ is suppressed from the notation. On the support Ω_n of χ, (C.33) holds. Moreover, by (C.27) and (C.28), we also have this crude bound

$$\|\mathcal{D}^{ij(x)}\| = \|(E^{ij(x)} - L^{ij(x)})G^{ij(x)}\| \leqslant C.$$

Therefore,

$$|\mathcal{D}_{vw}^{ij(x)}|^{2p-m} \leqslant C^p, \qquad 1 \leqslant m \leqslant 4.$$

It follows from (C.33) that

$$\left(\mathbb{E}\left[\sup_{|x|\leqslant|E_{ij}|}\Big||v_i|\sigma_{ij}^2\partial_{ij}^3 f_{ij}(E^{ij(x)})\Big|^2\right]\right)^{1/2} \leqslant C^p\sum_{m=1}^4 p^{m-1}\frac{\sigma_{\max}^{m-1}}{|z|^{m+1}}.$$

Since $\sigma_{\max} \leqslant \sqrt{R_{\max}} \leqslant |z|/6$, we simplify this bound to

$$\left(\mathbb{E}\left[\sup_{|x|\leqslant|E_{ij}|}\Big||v_i|\sigma_{ij}^2\partial_{ij}^3 f_{ij}(E^{ij(x)})\Big|^2\right]\right)^{1/2} \leqslant C^p p^3\frac{1}{|z|^2}.$$

Combining the above bound and (C.34), the total contribution of Term 2 is bounded by

$$\begin{aligned}&16\sum_{i,j}|v_i|\left(\mathbb{E}\left[\sup_{|x|\leqslant|E_{ij}|}|\partial_{ij}^3 f_{ij}(E^{ij(x)})|^2\right]\right)^{1/2}\left(\mathbb{E}\left[|E_{ij}|^8\mathbf{1}_{|E_{ij}|>t}\right]\right)^{1/2}\\&\leqslant C^p p^3\exp\big(-10(D+6)\log^2 n\big)\frac{\theta^2}{|z|^2}.\end{aligned}$$

Finally, we arrive at

$$\begin{aligned}|\mathtt{R}_3| &\leqslant C\theta^2\sum_{m=1}^4 p^{m-1}\frac{\sigma_{\max}^{m-1}}{|z|^{m+1}}\mathbb{E}\left[\left(|\mathcal{D}_{vw}|+22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m}\right]+C^p p^3\exp\big(-10(D+6)\log^2 n\big)\frac{\theta^2}{|z|^2}\\&=C(K\beta)^2\sum_{m=1}^4 p^{m-1}\frac{\sigma_{\max}^{m+1}}{|z|^{m+1}}\mathbb{E}\left[\left(|\mathcal{D}_{vw}|+22\frac{\mathtt{L}_n}{|z|}\right)^{2p-m}\right]+C^p p^3\exp\big(-10(D+6)\log^2 n\big)\frac{\theta^2}{|z|^2}.\end{aligned}$$

The bounded sparse-entry regime follows the same approach. Indeed, the Term 2 in the bound for $|\mathtt{R}_3|$ is equal to zero because $\max_{i,j}|E_{ij}|\leqslant \mathtt{L}_n$ almost surely. This simplifies the derivation. We skip the details. The proof of Lemma C.3 is now complete.

D Proof of Proposition 8.2

Recall that

$$H=\operatorname{diag}(h_1,\dots,h_n),\qquad h_i=\sum_{j=1}^n\sigma_{ij}^2\phi_j;$$
$$L=\operatorname{diag}(l_1,\dots,l_n),\qquad l_i=\sum_{j=1}^n\sigma_{ij}^2 G_{jj}.$$

Set

$$\mathcal{D}:=(E-L)G,\qquad \mathcal{D}_{ii}:=e_i^{\mathrm{T}}\mathcal{D}e_i.$$

Note that $\max_i|\phi_i(z)|\leqslant\frac{3}{2|z|}$ by Lemma 5.1. Define

$$d_i:=\phi_i-G_{ii},\qquad d:=(d_1,\dots,d_n)^{\mathrm{T}}$$

We first bound $\|d\|_\infty$. From $(zI-E)G=I$, we have $EG=zG-I$. Therefore, taking the (i,i)-entry of $\mathcal{D}=(E-L)G$ yields

$$\mathcal{D}_{ii}=(EG)_{ii}-l_iG_{ii}=(z-l_i)G_{ii}-1.$$

Equivalently,

$$(z-l_i)G_{ii}=1+\mathcal{D}_{ii}.$$

On the other hand, we have from the QVE that $(z-h_i)\phi_i=1$. Thus

$$\begin{aligned}(z-h_i)(\phi_i-G_{ii})&=(z-h_i)\phi_i-(z-h_i)G_{ii}\\&=1-(z-h_i)G_{ii}\\&=(z-l_i)G_{ii}-\mathcal{D}_{ii}-(z-h_i)G_{ii}\\&=(h_i-l_i)G_{ii}-\mathcal{D}_{ii}.\end{aligned}$$

Multiplying by ϕ_i, since $\phi_i = (z - h_i)^{-1}$, yields

$$d_i = \phi_i G_{ii}(h_i - l_i) - \phi_i \mathcal{D}_{ii}. \tag{D.1}$$

Moreover,

$$h_i - l_i = \sum_{j=1}^{n} \sigma_{ij}^2 (\phi_j - G_{jj}) = \sum_{j=1}^{n} \sigma_{ij}^2 d_j.$$

Substituting this into (D.1), we obtain

$$d_i = \phi_i G_{ii} \Big(\sum_{j=1}^{n} \sigma_{ij}^2 \Big) d_j - \phi_i \mathcal{D}_{ii}.$$

Thus, by taking the ℓ_∞ norm, we get

$$\|d\|_\infty \leqslant \max_i |\phi_i G_{ii}| \cdot R_{\max} \|d\|_\infty + \max_i |\phi_i| \cdot \max_i |\mathcal{D}_{ii}|.$$

Since

$$\max_i |\phi_i G_{ii}| \leqslant \frac{3}{2|z|} \cdot \frac{2}{|z|} = \frac{3}{|z|^2},$$

we bound

$$\|d\|_\infty \leqslant \frac{3R_{\max}}{|z|^2} \|d\|_\infty + \frac{3}{2|z|} \max_i |\mathcal{D}_{ii}|.$$

Since $|z|^2 \geqslant 36 R_{\max}$ by our assumption, we further obtain

$$\|d\|_\infty \leqslant \frac{2}{|z|} \max_i |\mathcal{D}_{ii}|. \tag{D.2}$$

We now return to $v^{\mathrm{T}}(L - H)Gw$. Since v, w are unit vectors,

$$|v^{\mathrm{T}}(L - H)Gw| \leqslant \|L - H\| \cdot \|G\|,$$

where

$$\|L - H\| = \max_i \Big| \sum_{j=1}^{n} \sigma_{ij}^2 (G_{jj} - \phi_j) \Big| \leqslant R_{\max} \|d\|_\infty.$$

Combining this with $\|G\| \leqslant 2/|z|$ and (D.2), we arrive at

$$|v^{\mathrm{T}}(L - H)Gw| \leqslant \frac{4R_{\max}}{|z|^2} \max_i |\mathcal{D}_{ii}| \leqslant \frac{1}{9} \max_i |\mathcal{D}_{ii}|. \tag{D.3}$$

To bound $\mathcal{D}_{ii}$, we apply Proposition 8.1 with $v = w = e_i$ and take a union bound to obtain

$$\max_{1 \leqslant i \leqslant n} |\mathcal{D}_{ii}| \leqslant C \left(\frac{\mathfrak{m}_D}{|z|} + n^{-500} \right)$$

with probability at least $1 - n^{-D-4}$. In the bounded sparse-entry regime, the analogous bound follows without the n^{-500} term.

Combining the above estimate with (D.3) proves Proposition 8.2.

E Proof of Corollary 2

Write
$$\Xi(z) = G(z) - \Phi(z).$$
We start with the following consequence of the isotropic local law:

COROLLARY 3 (Compressed local law on the signal subspace)**.** *Fix $D > 0$. Suppose the assumptions of Theorem 4 hold. Then, for every fixed $z \in \mathbb{C}_+$ with $|z| \geqslant 6\sqrt{R_{\max}}$, with probability at least $1 - n^{-D-2}$,*
$$\left\|U^{\mathrm{T}}(G(z) - \Phi(z))U\right\| \leqslant Cr\frac{\mathfrak{m}_D}{|z|^2}.$$

Proof of Corollary 3. Since
$$\begin{aligned}\left\|U^{\mathrm{T}}(G(z) - \Phi(z))U\right\| \leqslant \left\|U^{\mathrm{T}}(G(z) - \Phi(z))U\right\|_F &= \sqrt{\sum_{i,j=1}^{r}\left|e_i{}^{\mathrm{T}}U^{\mathrm{T}}(G(z) - \Phi(z))Ue_j\right|^2} \\ &\leqslant r\max_{1\leqslant i,j\leqslant r}\left|e_i{}^{\mathrm{T}}U^{\mathrm{T}}(G(z) - \Phi(z))Ue_j\right|,\end{aligned}$$
the conclusion of Corollary 3 follows directly from Theorem 4 and the union bounds. □

To extend the pointwise bound Corollary 3 to the spectral domain $\mathcal{S}_{\mathrm{out}} := \{z \in \mathbb{C} : 6\sqrt{R_{\max}} \leqslant |z| \leqslant 2R_{\max}^3\}$, we use a standard ε-net argument. Set
$$\eta := \min\left\{1, \frac{r\mathfrak{m}_D}{20}\right\}.$$
Let $\mathcal{N}_\eta$ be a η-net of $\mathcal{S}_{\mathrm{out}}$. A simple volume argument (see for instance [74, Lemma 3.3]) shows that
$$|\mathcal{N}_\eta| \leqslant \left(1 + \frac{8R_{\max}^3}{\eta}\right)^2.$$
Since $R_{\max} \leqslant n^{100}$ and $\sigma_{\max}^{-1} \leqslant n^{100}$, we have $\mathfrak{m}_D \geqslant cn^{-100}$ for an absolute constant $c > 0$. Thus $R_{\max}^3/\eta \leqslant C'n^{400}$ and $|\mathcal{N}_\eta| \leqslant Cn^{800}$.

Applying Corollary 3 with $D_1 = D + 810$ and taking a union bound over $z \in \mathcal{N}_\eta$, we obtain
$$\mathbb{P}\left(\max_{z\in\mathcal{N}_\eta}|z|^2\left\|U^{\mathrm{T}}\Xi(z)U\right\| \geqslant Cr\mathfrak{m}_{D_1}\right) \leqslant |\mathcal{N}_\eta|n^{-D-814} < n^{-D-10}.$$
Using $\mathfrak{m}_{D_1} = \mathfrak{m}_{D+810} \leqslant C'\mathfrak{m}_D$, we establish that with probability at least $1 - n^{-D-10}$,
$$\max_{z\in\mathcal{N}_\eta}|z|^2\left\|U^{\mathrm{T}}\Xi(z)U\right\| \leqslant Cr\mathfrak{m}_D.$$

Now we extend the estimate from $\mathcal{N}_\eta$ to the whole domain $\mathcal{S}_{\mathrm{out}}$. Define
$$f(z) := z^2U^{\mathrm{T}}G(z)U, \qquad g(z) := z^2U^{\mathrm{T}}\Phi(z)U.$$
We first show that f and g are Lipschitz on $\mathcal{S}$.

Let $z, w \in \mathcal{S}_{\text{out}}$ and assume without loss of generality that $|z| \geqslant |w| \geqslant 6\sqrt{R_{\max}}$. By the resolvent identity,

$$G(z) - G(w) = (w - z)G(z)G(w).$$

Using $\|G(z)\| \leqslant 2/|z|$ and $\|G(w)\| \leqslant 2/|w|$, we get

$$\begin{aligned}\|f(z) - f(w)\| &\leqslant |z||z - w|\|G(z)\| + |w||z - w|\|G(z)\| \\ &\quad + |w|^2\|G(z) - G(w)\| \\ &\leqslant 2|z - w| + 2\frac{|w|}{|z|}|z - w| + 4\frac{|w|}{|z|}|z - w| \\ &\leqslant 8|z - w|.\end{aligned}$$

Hence f is 8-Lipschitz on $\mathcal{S}$.

Next we prove the Lipschitz bound for g. For each $i \in [n]$, the QVE gives

$$\frac{1}{\phi_i(z)} = z - \sum_j \sigma_{ij}^2 \phi_j(z), \qquad \frac{1}{\phi_i(w)} = w - \sum_j \sigma_{ij}^2 \phi_j(w).$$

Subtracting the two identities and multiplying by $\phi_i(z)\phi_i(w)$, we obtain

$$\phi_i(z) - \phi_i(w) = \phi_i(z)\phi_i(w)\left[w - z + \sum_j \sigma_{ij}^2(\phi_j(z) - \phi_j(w))\right].$$

Therefore, by Lemma 5.1,

$$\|\Phi(z) - \Phi(w)\| \leqslant \frac{9}{4|z||w|}\left(|z - w| + R_{\max}\|\Phi(z) - \Phi(w)\|\right).$$

Since $|z||w| \geqslant 36R_{\max}$, we have

$$\frac{9R_{\max}}{4|z||w|} \leqslant \frac{1}{16}.$$

Thus,

$$\|\Phi(z) - \Phi(w)\| \leqslant \frac{12}{5}\frac{|z - w|}{|z||w|}.$$

Using again Lemma 5.1, we have $\|\Phi(z)\| \leqslant 3/(2|z|)$. Hence

$$\begin{aligned}\|g(z) - g(w)\| &\leqslant |z||z - w|\|\Phi(z)\| + |w||z - w|\|\Phi(z)\| \\ &\quad + |w|^2\|\Phi(z) - \Phi(w)\| \\ &\leqslant \frac{3}{2}|z - w| + \frac{3}{2}\frac{|w|}{|z|}|z - w| + \frac{12}{5}\frac{|w|}{|z|}|z - w| \\ &< 6|z - w|.\end{aligned}$$

Therefore g is 6-Lipschitz on $\mathcal{S}$.

Now take any $z \in \mathcal{S}_{\rm out}$. By the definition of the η-net, there exists $w \in \mathcal{N}_\eta$ such that $|z - w| \leqslant \eta$. Then

$$\begin{aligned} |z|^2 \left\| U^{\rm T} \Xi(z) U \right\| &= \|f(z) - g(z)\| \\ &\leqslant \|f(z) - f(w)\| + \|f(w) - g(w)\| + \|g(w) - g(z)\| \\ &\leqslant 14\eta + |w|^2 \left\| U^{\rm T} \Xi(w) U \right\|. \end{aligned}$$

Since $\eta \leqslant \frac{r\mathfrak{m}_D}{20}$, we get

$$\sup_{z \in \mathcal{S}_{\rm out}} |z|^2 \left\| U^{\rm T} \Xi(z) U \right\| \leqslant \frac{14 r\mathfrak{m}_D}{20} + Cr\mathfrak{m}_D \leqslant C' r\mathfrak{m}_D$$

with probability at least $1 - n^{-D-10}$. This completes the proof of (6.1).

The proof of (6.2) is identical to the proof of (6.1). One applies Theorem 4 to the pairs (e_i, u_a) for $i \in [n]$ and $a \in [r]$, takes a union bound over i, a, and uses the same net argument over $\mathcal{S}_{\rm out}$.

Finally,

$$e_i^{\rm T} Q \Xi(z) U = e_i^{\rm T} \Xi(z) U - e_i^{\rm T} U \cdot U^{\rm T} \Xi(z) U,$$

so (6.3) follows from (6.2) and (6.1).

F Proofs of (2.3) and (2.4)

We provide the proofs of (2.3) and (2.4) below.

Proof of (2.3): First, by [25, Exercises VII. 1. 9 1.11],

$$\| \sin \angle(U_{l:s}, \widetilde{U}_{l:s}) \| = \|(I - P_{U_{l:s}}) \widetilde{U}_{l:s}\| = \|P_{U_{l:s}^\perp} \widetilde{U}_{l:s}\|.$$

From the decomposition $P_{U_{l:s}} = P_{U_s} - P_{U_{l-1}}$, we can express

$$P_{U_{l:s}^\perp} = I - P_{U_{l:s}} = (I - P_{U_s}) + P_{U_{l-1}} = P_{U_s^\perp} + P_{U_{l-1}}.$$

It follows that

$$\|P_{U_{l:s}^\perp} \widetilde{U}_{l:s}\| \leqslant \|P_{U_s^\perp} \widetilde{U}_{l:s}\| + \|P_{U_{l-1}} \widetilde{U}_{l:s}\|. \tag{F.1}$$

For the first term, observe that the columns of $\widetilde{U}_{l:s}$ are a subset of the columns of $\widetilde{U}_s$. Thus

$$\|P_{U_s^\perp} \widetilde{U}_{l:s}\| \leqslant \|P_{U_s^\perp} \widetilde{U}_s\| = \| \sin \angle(U_s, \widetilde{U}_s)\|.$$

For the second term in (F.1), since $\widetilde{U}_{l:s}$ is orthogonal to $\widetilde{U}_{l-1}$, we have $P_{\widetilde{U}_{l-1}^\perp} \widetilde{U}_{l:s} = \widetilde{U}_{l:s}$. Then

$$P_{U_{l-1}} \widetilde{U}_{l:s} = P_{U_{l-1}} P_{\widetilde{U}_{l-1}^\perp} \widetilde{U}_{l:s},$$

and therefore,

$$\|P_{U_{l-1}} P_{\widetilde{U}_{l-1}^\perp} \widetilde{U}_{l:s}\| \leqslant \|P_{U_{l-1}} P_{\widetilde{U}_{l-1}^\perp}\| \cdot \|\widetilde{U}_{l:s}\| = \| \sin \angle(U_{l-1}, \widetilde{U}_{l-1})\|.$$

Substituting these two bounds back into (F.1) completes the proof of (2.3).

Proof of (2.4): Let $U_{l:s}^{\mathrm{T}}\widetilde{U}_{l:s} = WDV^{\mathrm{T}}$ be the singular value decomposition. We choose $O = WV^{\mathrm{T}}$ and then decompose

$$\begin{aligned}\widetilde{U}_{l:s} - U_{l:s}O &= (I - P_{U_{l:s}})\widetilde{U}_{l:s} + P_{U_{l:s}}(\widetilde{U}_{l:s} - U_{l:s}O) \\ &= (I - P_{U_{l:s}})\widetilde{U}_{l:s} + U_{l:s}(U_{l:s}^{\mathrm{T}}\widetilde{U}_{l:s} - O).\end{aligned}$$

Taking the $2 \to \infty$ norm and applying the triangle inequality yields

$$\|\widetilde{U}_{l:s} - U_{l:s}O\|_{2,\infty} \leqslant \|(I - P_{U_{l:s}})\widetilde{U}_{l:s}\|_{2,\infty} + \|U_{l:s}(U_{l:s}^{T}\widetilde{U}_{l:s} - O)\|_{2,\infty}. \tag{F.2}$$

By our choice of O, the second term in (F.2) is bounded by

$$\|U_{l:s}W(D - I)V^{\mathrm{T}}\|_{2,\infty} \leqslant \|U_{l:s}\|_{2,\infty}\|W(I - D)V^{\mathrm{T}}\| = \|U_{l:s}\|_{2,\infty}\|I - D\|.$$

Note that the diagonal entries of D are the cosines of the principal angles θ_i. The entries of $I - D^2$ are exactly $\sin^2\theta_i$. Because $D \succeq 0$, we have $I - D \preceq I - D^2$, which implies

$$\|I - D\| \leqslant \|I - D^2\| = \|\sin\angle(U_{l:s}, \widetilde{U}_{l:s})\|^2.$$

This proves the main inequality in (2.4). It remains to bound $\|(I - P_{U_{l:s}})\widetilde{U}_{l:s}\|_{2,\infty}$. Using $I - P_{U_{l:s}} = (I - P_{U_s}) + P_{U_{l-1}}$, we expand

$$(I - P_{U_{l:s}})\widetilde{U}_{l:s} = (I - P_{U_s})\widetilde{U}_{l:s} + P_{U_{l-1}}\widetilde{U}_{l:s}.$$

Thus

$$\|(I - P_{U_{l:s}})\widetilde{U}_{l:s}\|_{2,\infty} \leqslant \|(I - P_{U_s})\widetilde{U}_{l:s}\|_{2,\infty} + \|P_{U_{l-1}}\widetilde{U}_{l:s}\|_{2,\infty}. \tag{F.3}$$

For the first term, since $(I - P_{U_s})\widetilde{U}_{l:s}$ consists of a subset of the columns of $(I - P_{U_s})\widetilde{U}_s$, we get

$$\|(I - P_{U_s})\widetilde{U}_{l:s}\|_{2,\infty} \leqslant \|(I - P_{U_s})\widetilde{U}_s\|_{2,\infty}.$$

For the second term, plugging in $P_{U_{l-1}} = U_{l-1}U_{l-1}^{\mathrm{T}}$, we find

$$\|P_{U_{l-1}}\widetilde{U}_{l:s}\|_{2,\infty} = \|U_{l-1}(U_{l-1}^{\mathrm{T}}\widetilde{U}_{l:s})\|_{2,\infty} \leqslant \|U_{l-1}\|_{2,\infty}\|U_{l-1}^{\mathrm{T}}\widetilde{U}_{l:s}\|.$$

Finally, because $\widetilde{U}_{l:s}$ is orthogonal to $\widetilde{U}_{l-1}$, we obtain

$$\|U_{l-1}^{\mathrm{T}}\widetilde{U}_{l:s}\| = \|U_{l-1}^{\mathrm{T}}P_{\widetilde{U}_{l-1}^{\perp}}\widetilde{U}_{l:s}\| \leqslant \|U_{l-1}^{\mathrm{T}}P_{\widetilde{U}_{l-1}^{\perp}}\|\|\widetilde{U}_{l:s}\| = \|\sin\angle(U_{l-1}, \widetilde{U}_{l-1})\|.$$

Substituting these bounds into (F.3) finishes the proof.

G Extension to rectangular matrices: singular subspace perturbations

We briefly discuss the extension of current results to the rectangular model:

$$\widetilde{X} = X + E \in \mathbb{R}^{n_1 \times n_2},$$

where $X = U\Sigma V^{\mathrm{T}} = \sum_{i=1}^{r} \sigma_i u_i v_i^{\mathrm{T}}$ is a rank r matrix with singular values $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_r > 0$. The random noise matrix E has i.i.d. entries with $\mathbb{E}(E_{ij}) = 0$ and $\mathbb{E}(E_{ij}^2) = \sigma_{ij}^2$. Denote $\Sigma = (\sigma_{ij}^2) \in \mathbb{R}^{n_1 \times n_2}$. The entries of E are sub-Gaussian with a variance profile: $\|E_{ij}\|_{\psi_2} \leqslant K\sigma_{ij}$.

Using the standard Hermitian dilation (or linearization), we embed the rectangular matrices into larger symmetric block matrices by considering the $(n_1 + n_2) \times (n_1 + n_2)$ augmented symmetric matrices:

$$\mathbf{A} = \begin{pmatrix} 0 & X \\ X^{\mathrm{T}} & 0 \end{pmatrix}, \qquad \mathbf{E} = \begin{pmatrix} 0 & E \\ E^{\mathrm{T}} & 0 \end{pmatrix}, \quad \widetilde{\mathbf{A}} = \mathbf{A} + \mathbf{E}.$$

The nonzero eigenvalues of $\mathbf{A}$ are

$$\lambda_1 = \sigma_1 \geqslant \cdots \geqslant \lambda_r = \sigma_r > 0 > \lambda_{r+1} = -\sigma_1 \geqslant \cdots \geqslant \lambda_{2r} = -\sigma_r$$

with corresponding eigenvectors

$$w_j := \frac{1}{\sqrt{2}} \begin{pmatrix} u_j \\ v_j \end{pmatrix}, \qquad w_{j+r} := \frac{1}{\sqrt{2}} \begin{pmatrix} u_j \\ -v_j \end{pmatrix}, \qquad 1 \leqslant j \leqslant r.$$

Let $\mathbf{A} = WDW^{\mathrm{T}}$, where

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} U & U \\ V & -V \end{pmatrix}.$$

For $I \subseteq [2r]$, we denote by W_I the submatrix of W formed by columns $\{w_i : i \in I\}$. All information about the top-k singular subspaces U_k and V_k is encoded in the eigenspace W_{I_k}, where

$$I_k := \{1, \ldots, k\} \cup \{k+1, \ldots, k+r\}.$$

Hence, we can directly translate our eigenspace perturbation results to singular subspaces. We now introduce the relevant parameters. Define

$$R_i^{(L)} := \sum_{j=1}^{n_1} \sigma_{ij}^2, \qquad R_j^{(R)} := \sum_{i=1}^{n_2} \sigma_{ij}^2,$$

and let $\mathcal{R}_L := \mathrm{diag}(R_i^{(L)})_{1 \leqslant i \leqslant n_1}$, $\mathcal{R}_R := \mathrm{diag}(R_i^{(R)})_{1 \leqslant i \leqslant n_2}$,

$$\mathcal{R}_{\mathrm{aug}} := \mathrm{diag}(\mathcal{R}_L, \mathcal{R}_R), \qquad R_{\max}^{\mathrm{aug}} := \max\Big\{\max_i R_i^{(L)}, \max_j R_j^{(R)}\Big\}.$$

Define $M_{\mathrm{aug}} \equiv M_{\mathrm{aug,D}}$ as in (2.1) with the dimension factor $n_1 + n_2$ and set

$$\rho_k^{\mathrm{aug}} := 10 M_{\mathrm{aug}} + 15 \frac{\mathrm{osc}(\mathcal{R}_{\mathrm{aug}})}{\sigma_k},$$

and

$$\delta_k := \sigma_k - \sigma_{k+1}.$$

Also, we denote $\mathcal{V} = W^{\mathrm{T}}\mathcal{R}_{\mathrm{aug}}W$. Note that

$$\mathcal{V}_{\mathcal{J}\mathcal{I}} = W_{\mathcal{J}}^{\mathrm{T}}\mathcal{R}_{\mathrm{aug}}W_{\mathcal{I}} = \frac{1}{2}\left(U_{\mathcal{J}}^{\mathrm{T}}\mathcal{R}_L U_{\mathcal{I}} + V_{\mathcal{J}}^{\mathrm{T}}\mathcal{R}_R V_{\mathcal{I}}\right).$$

We define $\mathcal{B}_k^{\mathrm{aug}}$ similar to $\mathcal{B}_k$ (2.2):

$$\mathcal{B}_k^{\mathrm{aug}} := \frac{10\sqrt{k}}{\delta_k\sigma_k}\left(\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| + 8\frac{R_{\max}^{\mathrm{aug}}}{\sigma_k^2}\mathrm{osc}(\mathcal{R}_{\mathrm{aug}})\right) + 4\sqrt{k}\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\sigma_k^2}.$$

We can then directly translate eigenspace perturbation results from the symmetric model $\widetilde{\mathbf{A}} = \mathbf{A} + \mathbf{E}$ to the rectangular model $\widetilde{X} = X + E$ using the revised parameters above and the following key relationships (see [83] for details): for any unitarily invariant norm $|||\cdot|||$,

$$\max\{|||\sin\angle(U_k, \widetilde{U}_k)|||, |||\sin\angle(V_k, \widetilde{V}_k)|||\} \leqslant |||\sin\angle(W_I, \widetilde{W}_k)|||$$

and

$$\max\{\|\widetilde{U}_k - P_{U_k}\widetilde{U}_k\|_{2,\infty}, \|\widetilde{V}_k - P_{V_k}\widetilde{V}_k\|_{2,\infty}\} \leqslant \|\widetilde{W}_I - P_{W_I}\widetilde{W}_I\|_{2,\infty}.$$

Specifically, this implies that the following $2 \to \infty$ bound holds with probability at least $1 - n^{-D}$:

$$\begin{aligned}&\min_{O\in\mathbb{O}(k)} \|\widetilde{U}_k - U_k O\|_{2,\infty} \\ &\quad\leqslant 4\max\{\|U\|_{2,\infty}, \|V\|_{2,\infty}\}\left(\mathcal{B}_k^{\mathrm{aug}} + 30\frac{R_{\max}^{\mathrm{aug}}}{\sigma_k^2}\right) + 120\sqrt{k}\frac{M_{\mathrm{aug}}}{\sigma_k}.\end{aligned} \tag{G.1}$$

The same bound also holds for $\min_{O\in\mathbb{O}(k)}\|\widetilde{V}_k - V_k O\|_{2,\infty}$.

We conclude by clarifying the nature of the coupled terms in (G.1). While the dependence on both variance profiles ($\mathcal{R}_L$ and $\mathcal{R}_R$) reflects the intrinsic statistical coupling in the asymmetric noise matrix, the dependence on the maximum joint incoherence $\max\{\|U\|_{2,\infty}, \|V\|_{2,\infty}\}$ and the total dimension $n_1 + n_2$ is an artifact of the symmetric embedding framework. We plan to address this limitation in future work.

H Proofs of Lemma 3.1, Proposition 3.1, (I.1) and (I.2)

H.1 Proof of Lemma 3.1

For brevity, write

$$L_{\mathcal{J},s} := I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}}.$$

For $j \in \mathcal{J}$, set

$$\alpha_j(z) := \lambda_j^{-1} - u_j^{\mathrm{T}}\Phi(z)u_j.$$

Recall the quantity defined in Proposition 7.2

$$\Delta^{\Phi}_{\mathcal{J}}(\widetilde{\lambda}_s) := \widetilde{\lambda}_s \min_{j\in\mathcal{J}} \left|1 - \lambda_j u_j^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) u_j\right| = \min_{j\in\mathcal{J}} \left|\widetilde{\lambda}_s \lambda_j \alpha_j(\widetilde{\lambda}_s)\right|.$$

We first relate $s_{\min}(L_{\mathcal{J},s})$ to $\Delta^{\Phi}_{\mathcal{J}}(\widetilde{\lambda}_s)$. Decompose

$$U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{J}} = \operatorname{diag}\big(u_j^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) u_j\big)_{j\in\mathcal{J}} + Q_{\mathcal{J}}(\widetilde{\lambda}_s),$$

where

$$Q_{\mathcal{J}}(\widetilde{\lambda}_s) := \operatorname{Off}\big(U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{J}}\big).$$

Then

$$L_{\mathcal{J},s} = \operatorname{diag}\left(1 - \lambda_j u_j^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) u_j\right)_{j\in\mathcal{J}} - Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}.$$

Hence

$$s_{\min}(L_{\mathcal{J},s}) \geqslant \frac{\Delta^{\Phi}_{\mathcal{J}}(\widetilde{\lambda}_s)}{\widetilde{\lambda}_s} - \|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}\|. \tag{H.1}$$

We next estimate the off-diagonal term $\|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}\|$. Using the expansion

$$\Phi(z) = \frac{1}{z} I + \frac{1}{z^3} R + \varepsilon(z),$$

we have

$$Q_{\mathcal{J}}(\widetilde{\lambda}_s) = \frac{1}{\widetilde{\lambda}_s^3} \operatorname{Off}(\mathcal{V}_{\mathcal{J}\mathcal{J}}) + \operatorname{Off}\big(U_{\mathcal{J}}^{\mathrm{T}} \varepsilon(\widetilde{\lambda}_s) U_{\mathcal{J}}\big).$$

Since the eigenvalues in $\mathcal{J}$ are positive and bounded above by λ_{k+1},

$$\|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}\| \leqslant \frac{\lambda_{k+1}}{\widetilde{\lambda}_s^3} \|\operatorname{Off}(\mathcal{V}_{\mathcal{J}\mathcal{J}})\| + \lambda_{k+1} \left\|\operatorname{Off}\big(U_{\mathcal{J}}^{\mathrm{T}} \varepsilon(\widetilde{\lambda}_s) U_{\mathcal{J}}\big)\right\|.$$

By Lemma 5.2,

$$\left\|\operatorname{Off}\big(U_{\mathcal{J}}^{\mathrm{T}} \varepsilon(\widetilde{\lambda}_s) U_{\mathcal{J}}\big)\right\| \leqslant \frac{1}{2} \operatorname{osc}(\varepsilon(\widetilde{\lambda}_s)) \leqslant \frac{15}{2} \frac{R_{\max}}{\widetilde{\lambda}_s^5} \operatorname{osc}(\mathcal{R}).$$

Therefore

$$\|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}\| \leqslant \frac{\lambda_{k+1}}{\widetilde{\lambda}_s^3} \|\operatorname{Off}(V_{\mathcal{J}\mathcal{J}})\| + \frac{15}{2} \frac{\lambda_{k+1} R_{\max}}{\widetilde{\lambda}_s^5} \operatorname{osc}(\mathcal{R}). \tag{H.2}$$

We now verify that the off-diagonal term in (H.1) is small compared with the diagonal part. Using $\widetilde{\lambda}_s \geqslant \frac{1}{2}\lambda_k$, $R_{\max} \leqslant 36\lambda_k^2$, and $\|\operatorname{Off}(\mathcal{V}_{\mathcal{J}\mathcal{J}})\| \leqslant \operatorname{osc}(\mathcal{R})$, from (H.2), we get

$$\widetilde{\lambda}_s \|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}\| \leqslant \left(8 + \frac{5}{3}\right) \frac{\lambda_{k+1}}{\lambda_k^2} \operatorname{osc}(\mathcal{R}) \leqslant \frac{1}{20} \left(8 + \frac{5}{3}\right) \delta_k,$$

where the last inequality again follows from the gap assumption. Since $\Delta^{\Phi}_{\mathcal{J}}(\widetilde{\lambda}_s) \geqslant \delta_k/2$, we get

$$\|Q_{\mathcal{J}}(\widetilde{\lambda}_s)\Lambda_{\mathcal{J}}\| \leqslant \frac{1}{10} \left(8 + \frac{5}{3}\right) \frac{\Delta^{\Phi}_{\mathcal{J}}(\widetilde{\lambda}_s)}{\widetilde{\lambda}_s}.$$

Returning to (H.1), we obtain

$$s_{\min}(L_{\mathcal{J},s}) \geqslant \frac{1}{30}\frac{\Delta_{\mathcal{J}}^{\Phi}(\widetilde{\lambda}_s)}{\widetilde{\lambda}_s} \geqslant \frac{1}{60}\frac{\delta_k}{\widetilde{\lambda}_s}.$$

Thus $\|L_{\mathcal{J},s}^{-1}\| \leqslant 60\frac{\widetilde{\lambda}_s}{\delta_k}$. This proves the desired bound.

H.2 Proof of Proposition 3.1

For $s \in [k]$, the eigen-equation gives

$$\widetilde{u}_s = G(\widetilde{\lambda}_s)A\widetilde{u}_s = \Phi(\widetilde{\lambda}_s)A\widetilde{u}_s + \Xi(\widetilde{\lambda}_s)A\widetilde{u}_s,$$

where $\Xi(z) := G(z) - \Phi(z)$. It follows that

$$U^{\mathrm{T}}\widetilde{u}_s = U^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U\Lambda U^{\mathrm{T}}\widetilde{u}_s + U^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\Lambda U^{\mathrm{T}}\widetilde{u}_s.$$

Taking the $\mathcal{J}$-block yields

$$U_{\mathcal{J}}^{\mathrm{T}}\widetilde{u}_s = U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{I}}\Lambda_{\mathcal{I}} \cdot U_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s + U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}} \cdot U_{\mathcal{J}}^{\mathrm{T}}\widetilde{u}_s + U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\Lambda U^{\mathrm{T}}\widetilde{u}_s.$$

Collecting the $U_{\mathcal{J}}^{\mathrm{T}}\widetilde{u}_s$ terms gives

$$(I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}})U_{\mathcal{J}}^{\mathrm{T}}\widetilde{u}_s = U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{I}}\Lambda_{\mathcal{I}} \cdot U_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s + U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\Lambda U^{\mathrm{T}}\widetilde{u}_s.$$

Recall $\mathcal{T}_s$ from (3.1). We further get

$$U_{\mathcal{J}}^{\mathrm{T}}\widetilde{u}_s - \mathcal{T}_s U_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s = (I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}})^{-1}U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\Lambda U^{\mathrm{T}}\widetilde{u}_s. \tag{H.3}$$

By definition and $\mathcal{N} = \{r_+ + 1, \ldots, r\}$,

$$P_U\widetilde{u}_s = U_k U_k^{\mathrm{T}}\widetilde{u}_s + U_{\mathcal{J}}U_{\mathcal{J}}^{\mathrm{T}}\widetilde{u}_s + U_{\mathcal{N}}U_{\mathcal{N}}^{\mathrm{T}}\widetilde{u}_s.$$

Combining this with (H.3), we get

$$\begin{aligned} w_s^{\mathrm{orc}} &= P_U\widetilde{u}_s - U_{\mathcal{J}}\mathcal{T}_s U_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s \\ &= U_k U_k^{\mathrm{T}}\widetilde{u}_s + U_{\mathcal{N}}U_{\mathcal{N}}^{\mathrm{T}}\widetilde{u}_s + U_{\mathcal{J}}(I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}})^{-1}U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U \cdot \Lambda U^{\mathrm{T}}\widetilde{u}_s. \end{aligned} \tag{H.4}$$

Note that we have established in (7.6) that

$$\|U_{\mathcal{N}}^{\mathrm{T}}\widetilde{u}_s\| < 4\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 24\frac{R_{\max}}{\lambda_k^4}\mathrm{osc}(\mathcal{R}) + 4\frac{M}{\lambda_k}. \tag{H.5}$$

Now we estimate the last term on the right-hand side of (H.4). We first have

$$\|\Lambda U^{\mathrm{T}}\widetilde{u}_s\| \leqslant 2|\widetilde{\lambda}_s|.$$

To see this, from $(A + E)\widetilde{u}_s = \widetilde{\lambda}_s\widetilde{u}_s$, we have

$$A\widetilde{u}_s = (\widetilde{\lambda}_s I - E)\widetilde{u}_s.$$

Multiplying by U^{T} gives $\Lambda U^{\mathrm{T}}\widetilde{u}_s = U^{\mathrm{T}}A\widetilde{u}_s = U^{\mathrm{T}}(\widetilde{\lambda}_s I - E)\widetilde{u}_s$. Hence,

$$\|\Lambda U^{\mathrm{T}}\widetilde{u}_s\| \leqslant |\widetilde{\lambda}_s| + \|E\| \leqslant 2|\widetilde{\lambda}_s|$$

by our assumption $\lambda_k \geqslant 9\sqrt{R_{\max}}$ and Weyl's inequality.

On the event $\mathcal{E}_D$, we have

$$\|U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U\| \leqslant \frac{M}{\widetilde{\lambda}_s^2}.$$

Combining these estimates with Lemma 3.1 yields

$$\begin{aligned} &\|U_{\mathcal{J}}(I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}})^{-1}U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U \cdot \Lambda U^{\mathrm{T}}\widetilde{u}_s\| \\ &\quad\leqslant \|(I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}})^{-1}U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U \cdot \Lambda U^{\mathrm{T}}\widetilde{u}_s\| \leqslant 120\frac{M}{\delta_k}. \end{aligned} \tag{H.6}$$

From (H.4) and the orthogonality of the signal subspaces, $w_s^{\mathrm{orc}} - U_kU_k^{\mathrm{T}}\widetilde{u}_s$ lies entirely in the span of $U_k^{\perp}$. Combining the above bounds, we have

$$\begin{aligned} \|w_s^{\mathrm{orc}} - U_kU_k^{\mathrm{T}}\widetilde{u}_s\| &\leqslant \|U_{\mathcal{N}}^{\mathrm{T}}\widetilde{u}_s\| + \|U_{\mathcal{J}}(I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}})^{-1}U_{\mathcal{J}}^{\mathrm{T}}\Xi(\widetilde{\lambda}_s)U \cdot \Lambda U^{\mathrm{T}}\widetilde{u}_s\| \\ &\leqslant 4\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 24\frac{R_{\max}}{\lambda_k^4}\mathrm{osc}(\mathcal{R}) + 124\frac{M}{\delta_k}. \end{aligned}$$

Denote

$$\Delta^{\mathrm{orc}} := (w_1^{\mathrm{orc}} - U_kU_k^{\mathrm{T}}\widetilde{u}_1, \ldots, w_k^{\mathrm{orc}} - U_kU_k^{\mathrm{T}}\widetilde{u}_k).$$

It follows immediately that

$$\|\Delta^{\mathrm{orc}}\| \leqslant \|\Delta^{\mathrm{orc}}\|_F \leqslant \sqrt{k}\left(4\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 24\frac{R_{\max}}{\lambda_k^4}\mathrm{osc}(\mathcal{R}) + 124\frac{M}{\delta_k}\right) = \mathcal{E}_k^{\mathrm{osc}}.$$

We rewrite

$$W_k^{\mathrm{orc}} = (w_1^{\mathrm{orc}}, \ldots, w_k^{\mathrm{orc}}) = U_kU_k^{\mathrm{T}}\widetilde{U}_k + \Delta^{\mathrm{orc}}.$$

Note that

$$U_k^{\mathrm{orc}} = \mathrm{orth}(W_k^{\mathrm{orc}}) = W_k^{\mathrm{orc}}((W_k^{\mathrm{orc}})^{\mathrm{T}}W_k^{\mathrm{orc}})^{-1/2}. \tag{H.7}$$

We aim to bound

$$\|\sin\angle(U_k, U_k^{\mathrm{orc}})\| = \|P_{U_k^{\perp}}U_k^{\mathrm{orc}}\| = \|P_{U_k^{\perp}}W_k^{\mathrm{orc}}((W_k^{\mathrm{orc}})^{\mathrm{T}}W_k^{\mathrm{orc}})^{-1/2}\| \leqslant \frac{\|P_{U_k^{\perp}}W_k^{\mathrm{orc}}\|}{s_{\min}(W_k^{\mathrm{orc}})}.$$

First,

$$\|P_{U_k^{\perp}}W_k^{\mathrm{orc}}\| = \|P_{U_k^{\perp}}(U_kU_k^{\mathrm{T}}\widetilde{U}_k + \Delta^{\mathrm{orc}})\| = \|P_{U_k^{\perp}}\Delta^{\mathrm{orc}}\| \leqslant \|\Delta^{\mathrm{orc}}\|.$$

Next, since $W_k^{\mathrm{orc}} = U_kU_k^{\mathrm{T}}\widetilde{U}_k + \Delta^{\mathrm{orc}}$, and Δ^{orc} is orthogonal to U_k, we have

$$\sigma_{\min}(W_k^{\mathrm{orc}}) \geqslant \sigma_{\min}(U_kU_k^{\mathrm{T}}\widetilde{U}_k) = \sigma_{\min}(U_k^{\mathrm{T}}\widetilde{U}_k) = \sqrt{1 - \|\sin\angle(U_k, \widetilde{U}_k)\|^2}.$$

On the event that Theorem 3 holds, $\|\sin\angle(U_k, \widetilde{U}_k)\| \leqslant 1/100$. Thus,

$$\sigma_{\min}(W_k^{\mathrm{orc}}) \geqslant \frac{1}{2}.$$

We conclude that

$$\|\sin\angle(U_k, U_k^{\mathrm{orc}})\| \leqslant 2\|\Delta^{\mathrm{orc}}\| \leqslant 2\mathcal{E}_k^{\mathrm{osc}}.$$

For the $2 \to \infty$ norm bound, we start with the standard decomposition

$$\min_{O \in \mathbb{O}(k)} \|U_k^{\mathrm{orc}} - U_k O\|_{2,\infty} \leqslant \|(I - P_{U_k})U_k^{\mathrm{orc}}\|_{2,\infty} + \|U_k\|_{2,\infty}\|\sin\angle(U_k, U_k^{\mathrm{orc}})\|^2.$$

It remains to bound $\|(I - P_{U_k})U_k^{\mathrm{orc}}\|_{2,\infty}$. Since $P_{U_k^\perp} W_k^{\mathrm{orc}} = \Delta^{\mathrm{orc}}$, from (H.7), we get

$$(I - P_{U_k})U_k^{\mathrm{orc}} = \Delta^{\mathrm{orc}}((W_k^{\mathrm{orc}})^{\mathrm{T}} W_k^{\mathrm{orc}})^{-1/2}.$$

Thus

$$\begin{aligned}
\|(I - P_{U_k})U_k^{\mathrm{orc}}\|_{2,\infty} &\leqslant \|\Delta^{\mathrm{orc}}\|_{2,\infty}\|((W_k^{\mathrm{orc}})^{\mathrm{T}} W_k^{\mathrm{orc}})^{-1/2}\| \\
&\leqslant \frac{\|\Delta^{\mathrm{orc}}\|_{2,\infty}}{\sigma_{\min}(W_k^{\mathrm{orc}})} \leqslant 2\|\Delta^{\mathrm{orc}}\|_{2,\infty}.
\end{aligned}$$

We bound $\|\Delta^{\mathrm{orc}}\|_{2,\infty}$. From (H.4),

$$\begin{aligned}
\Delta^{\mathrm{orc}} &= (w_1^{\mathrm{orc}} - U_k U_k^{\mathrm{T}} \widetilde{u}_1, \ldots, w_k^{\mathrm{orc}} - U_k U_k^{\mathrm{T}} \widetilde{u}_k) \\
&= U_{\mathcal{N}} \cdot U_{\mathcal{N}}^{\mathrm{T}} \widetilde{U}_k + U_{\mathcal{J}} \cdot \mathtt{B}
\end{aligned}$$

where $\mathtt{B} := [\mathtt{b_1}, \cdots, \mathtt{b_k}]$ with columns

$$\mathtt{b}_s = (I - U_{\mathcal{J}}^{\mathrm{T}} \Phi(\widetilde{\lambda}_s) U_{\mathcal{J}} \Lambda_{\mathcal{J}})^{-1} U_{\mathcal{J}}^{\mathrm{T}} \Xi(\widetilde{\lambda}_s) U \cdot \Lambda U^{\mathrm{T}} \widetilde{u}_s.$$

In (H.6), we have

$$\|\mathtt{b}_s\| \leqslant 120\frac{M}{\delta_k}$$

and thus

$$\|\mathtt{B}\| \leqslant \|\mathtt{B}\|_F \leqslant 120\sqrt{k}\frac{M}{\delta_k}.$$

Also, by (H.5), we get

$$\begin{aligned}
\|U_{\mathcal{N}}^{\mathrm{T}} \widetilde{U}_k\| \leqslant \|U_{\mathcal{N}}^{\mathrm{T}} \widetilde{U}_k\|_F &\leqslant \sqrt{k} \max_{s \in k} \|U_{\mathcal{N}}^{\mathrm{T}} \widetilde{u}_s\| \\
&\leqslant 4\sqrt{k}\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 24\sqrt{k}\frac{R_{\max}}{\lambda_k^4}\mathrm{osc}(\mathcal{R}) + 4\sqrt{k}\frac{M}{\lambda_k}.
\end{aligned}$$

Now we get

$$\begin{aligned}
\|(I - P_{U_k})U_k^{\mathrm{orc}}\|_{2,\infty} &\leqslant 2\|\Delta^{\mathrm{orc}}\|_{2,\infty} = 2\max_{1 \leqslant i \leqslant n} \|e_i^{\mathrm{T}}(U_{\mathcal{N}} \cdot U_{\mathcal{N}}^{\mathrm{T}} \widetilde{U}_k + U_{\mathcal{J}} \cdot \mathtt{B})\| \\
&\leqslant 2\|U\|_{2,\infty}(\|U_{\mathcal{N}}^{\mathrm{T}} \widetilde{U}_k\| + \|\mathtt{B}\|) \\
&\leqslant \|U\|_{2,\infty}\left(8\sqrt{k}\frac{\|\mathcal{V}_{\mathcal{N}\mathcal{K}}\|}{\lambda_k^2} + 48\sqrt{k}\frac{R_{\max}}{\lambda_k^4}\mathrm{osc}(\mathcal{R}) + 248\sqrt{k}\frac{M}{\delta_k}\right) \\
&= 2\|U\|_{2,\infty}\mathcal{E}_k^{\mathrm{osc}}.
\end{aligned}$$

Finally, combining the bound on $\|\sin\angle(U_k, U_k^{\mathrm{orc}})\|$, we obtain

$$\begin{aligned}\min_{O\in\mathbb{O}(k)} \|U_k^{\mathrm{orc}} - U_k O\|_{2,\infty} &\leqslant \|(I-P_{U_k})U_k^{\mathrm{orc}}\|_{2,\infty} + \|U_k\|_{2,\infty}\|\sin\angle(U_k, U_k^{\mathrm{orc}})\|^2 \\ &\leqslant 3\|U\|_{2,\infty}\mathcal{E}_k^{\mathrm{osc}}.\end{aligned}$$

The proof is complete.

H.3 Proof of (I.1) and (I.2)

By the definitions of $\widehat{w}_s$ and w_s^{orc}, their difference is

$$\widehat{w}_s - w_s^{\mathrm{orc}} = P_{\widetilde{U}}\widetilde{u}_s - P_U\widetilde{u}_s - \left(\widetilde{U}_{\mathcal{J}}\widehat{\mathcal{T}}_s\widetilde{U}_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s - U_{\mathcal{J}}\mathcal{T}_s U_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s\right).$$

Since $P_{\widetilde{U}}\widetilde{u}_s = \widetilde{u}_s$, we get

$$\widehat{W}_k - W_k^{\mathrm{orc}} = (I-P_U)\widetilde{U}_k - \mathfrak{D}_k.$$

Let $\Delta := \|\widehat{W}_k - W_k^{\mathrm{orc}}\|$. Applying the triangle inequality, we bound

$$\Delta \leqslant \|(I-P_U)\widetilde{U}_k\| + \|\mathfrak{D}_k\| \leqslant 2\frac{\|E\|}{\lambda_k} + \|\mathfrak{D}_k\|.$$

The bound in the second inequality was established in (B.2).

Similarly, taking the $2\to\infty$ norm yields

$$\Delta_{2,\infty} := \|\widehat{W}_k - W_k^{\mathrm{orc}}\|_{2,\infty} \leqslant \|(I-P_U)\widetilde{U}_k\|_{2,\infty} + \|\mathfrak{D}_k\|_{2,\infty}.$$

Note that the assumption $\mathcal{E}_k^{\mathrm{orc}} + \frac{\|E\|}{\lambda_k} + \|\mathfrak{D}_k\| \leqslant c$ guarantees that $\mathcal{E}_k^{\mathrm{orc}} + \Delta \leqslant c \leqslant 1/10$ for a sufficiently small absolute constant c.

Recall from the proof of Proposition 3.1 that $W_k^{\mathrm{orc}} = U_k U_k^{\mathrm{T}}\widetilde{U}_k + \Delta^{\mathrm{orc}}$, where $\|P_{U_k^\perp}W_k^{\mathrm{orc}}\| \leqslant \|\Delta^{\mathrm{orc}}\| \leqslant \mathcal{E}_k^{\mathrm{orc}}$ and $\|P_{U_k^\perp}W_k^{\mathrm{orc}}\|_{2,\infty} \leqslant \|U\|_{2,\infty}\mathcal{E}_k^{\mathrm{orc}}$.

From the decomposition $\widehat{W}_k = W_k^{\mathrm{orc}} + (\widehat{W}_k - W_k^{\mathrm{orc}})$, we obtain

$$\|P_{U_k^\perp}\widehat{W}_k\| \leqslant \|P_{U_k^\perp}W_k^{\mathrm{orc}}\| + \|\widehat{W}_k - W_k^{\mathrm{orc}}\| \leqslant \mathcal{E}_k^{\mathrm{orc}} + \Delta.$$

Similarly, the $2\to\infty$ norm satisfies

$$\|P_{U_k^\perp}\widehat{W}_k\|_{2,\infty} \leqslant \|P_{U_k^\perp}W_k^{\mathrm{orc}}\|_{2,\infty} + \|\widehat{W}_k - W_k^{\mathrm{orc}}\|_{2,\infty} \leqslant \|U\|_{2,\infty}\mathcal{E}_k^{\mathrm{orc}} + \Delta_{2,\infty}. \tag{H.8}$$

Next, we lower-bound $s_{\min}(\widehat{W}_k)$. We established previously that $s_{\min}(W_k^{\mathrm{orc}}) \geqslant 1/2$. By Weyl's inequality,

$$s_{\min}(\widehat{W}_k) \geqslant s_{\min}(W_k^{\mathrm{orc}}) - \|\widehat{W}_k - W_k^{\mathrm{orc}}\| \geqslant \frac{1}{2} - \Delta \geqslant \frac{1}{2} - \frac{1}{10} = \frac{2}{5}$$

since $\Delta \leqslant 1/10$.

Similar to the proof of Proposition 3.1, we get

$$\|\sin\angle(U_k,\widehat{U}_k^{\mathrm{db}})\| = \|P_{U_k^\perp}\widehat{U}_k^{\mathrm{db}}\| \leqslant \frac{\|P_{U_k^\perp}\widehat{W}_k\|}{\sigma_{\min}(\widehat{W}_k)} \leqslant \frac{5}{2}(\mathcal{E}_k^{\mathrm{orc}}+\Delta).$$

For the $2\to\infty$ bound, we start with

$$\min_{O\in\mathbb{O}(k)}\|\widehat{U}_k^{\mathrm{db}}-U_kO\|_{2,\infty} \leqslant \|(I-P_{U_k})\widehat{U}_k^{\mathrm{db}}\|_{2,\infty}+\|U_k\|_{2,\infty}\|\sin\angle(U_k,\widehat{U}_k^{\mathrm{db}})\|^2.$$

Since $(I-P_{U_k})\widehat{U}_k^{\mathrm{db}} = P_{U_k^\perp}\widehat{W}_k(\widehat{W}_k^{\mathrm{T}}\widehat{W}_k)^{-1/2}$, we bound the first term using (H.8):

$$\|(I-P_{U_k})\widehat{U}_k^{\mathrm{db}}\|_{2,\infty} \leqslant \frac{\|P_{U_k^\perp}\widehat{W}_k\|_{2,\infty}}{\sigma_{\min}(\widehat{W}_k)} \leqslant \frac{5}{2}\left(\|U\|_{2,\infty}\mathcal{E}_k^{\mathrm{orc}}+\Delta_{2,\infty}\right).$$

Substituting this and the bound on $\|\sin\angle(U_k,\widehat{U}_k^{\mathrm{db}})\|^2$ back into the decomposition yields the desired bound.

I Plug-in de-biasing implementation and limitations

To keep the discussion simple, assume in this paragraph that r and r_+ are known and the variance profile Σ is known, so that $\Phi(z)$ is computable from the QVE..

Define

$$\widetilde{U}_{\mathcal{J}} := (\widetilde{u}_{k+1},\ldots,\widetilde{u}_{r_+}), \qquad \widetilde{U}_{\mathcal{I}} := (\widetilde{u}_1,\ldots,\widetilde{u}_k,\widetilde{u}_{r_++1},\ldots,\widetilde{u}_r),$$

with corresponding diagonal eigenvalue matrices $\widetilde{\Lambda}_{\mathcal{J}}$ and $\widetilde{\Lambda}_{\mathcal{I}}$. Also write

$$\widetilde{U} := (\widetilde{u}_1,\ldots,\widetilde{u}_r), \qquad P_{\widetilde{U}} := \widetilde{U}\widetilde{U}^{\mathrm{T}}.$$

If $\mathcal{J}=\varnothing$, no correction is needed and we set $\widehat{U}_k^{\mathrm{db}} := \widetilde{U}_k$. Otherwise, for $s\in[k]$, define

$$\widehat{\mathcal{T}}_s := \left(I-\widetilde{U}_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)\widetilde{U}_{\mathcal{J}}\widetilde{\Lambda}_{\mathcal{J}}\right)^{-1}\widetilde{U}_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)\widetilde{U}_{\mathcal{I}}\widetilde{\Lambda}_{\mathcal{I}},$$

whenever the inverse exists. The plug-in corrected vector is

$$\widehat{w}_s := P_{\widetilde{U}}\widetilde{u}_s - \widetilde{U}_{\mathcal{J}}\widehat{\mathcal{T}}_s\widetilde{U}_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s, \qquad s=1,\ldots,k.$$

Note that for $s\leqslant k$, $P_{\widetilde{U}}\widetilde{u}_s=\widetilde{u}_s$. Finally, set

$$\widehat{W}_k := (\widehat{w}_1,\ldots,\widehat{w}_k), \qquad \widehat{U}_k^{\mathrm{db}} := \mathrm{orth}(\widehat{W}_k).$$

To compare this estimator with the oracle estimator, define the correction discrepancy

$$\mathfrak{D}_k := \left(\widetilde{U}_{\mathcal{J}}\widehat{\mathcal{T}}_s\widetilde{U}_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s - U_{\mathcal{J}}\mathcal{T}_sU_{\mathcal{I}}^{\mathrm{T}}\widetilde{u}_s\right)_{s=1}^k.$$

Then the difference between the plug-in and oracle matrices is given by

$$\widehat{W}_k - W_k^{\mathrm{orc}} = (I - P_U)\widetilde{U}_k - \mathfrak{D}_k.$$

Consequently, the accuracy of the plug-in correction is governed by the norms of this difference. As shown in (B.2) below, $\|(I - P_U)\widetilde{U}_k\| \leqslant 2\|E\|/\lambda_k$. On the event that

$$\mathcal{E}_k^{\mathrm{orc}} + \frac{\|E\|}{\lambda_k} + \|\mathfrak{D}_k\| \leqslant c$$

for a sufficiently small absolute constant $c > 0$, the same argument as in Proposition 3.1 gives

$$\|\sin\angle(\widehat{U}_k^{\mathrm{db}}, U_k)\| \leqslant C\left(\mathcal{E}_k^{\mathrm{orc}} + \frac{\|E\|}{\lambda_k} + \|\mathfrak{D}_k\|\right). \tag{I.1}$$

Moreover,

$$\min_{O\in\mathbb{O}(k)} \|\widehat{U}_k^{\mathrm{db}} - U_k O\|_{2,\infty} \leqslant C\Big[\|U\|_{2,\infty}\mathcal{E}_k^{\mathrm{orc}} + \|(I - P_U)\widetilde{U}_k\|_{2,\infty} + \|\mathfrak{D}_k\|_{2,\infty} + \|U_k\|_{2,\infty}\left(\mathcal{E}_k^{\mathrm{orc}} + \frac{\|E\|}{\lambda_k} + \|\mathfrak{D}_k\|\right)^2\Big]. \tag{I.2}$$

The proofs of (I.1) and (I.2) are given in Appendix H.3 for completeness.

REMARK I.1 (Limitation and useful regimes)**.** *The plug-in correction $\widehat{U}_k^{\mathrm{db}}$ is not a universal replacement for the raw empirical eigenspace $\widetilde{U}_k$. The term $\mathfrak{D}_k$ measures the error in estimating the oracle correction from the empirical lower-positive $\mathcal{J}$-block. This term could be large in two situations. First, if the lower-positive block $\mathcal{J}$ contains weak or poorly separated spikes, then the empirical block $\widetilde{U}_{\mathcal{J}}$ may not estimate $U_{\mathcal{J}}$ accurately. Second, when eigenvalues are tightly clustered (δ_k is small), the matrices $I - U_{\mathcal{J}}^{\mathrm{T}}\Phi(\widetilde{\lambda}_s)U_{\mathcal{J}}\Lambda_{\mathcal{J}}$ become ill-conditioned. Inverting them will make $\|\mathfrak{D}_k\|$ blow up.*

Thus the plug-in estimator is useful only when

$$\|\mathfrak{D}_k\| \lesssim \mathcal{E}_k^{\mathrm{orc}}$$

and the removed bias is larger than the residual terms

$$\frac{\sqrt{k}}{\delta_k\lambda_k}\left(\|\mathcal{V}_{\mathcal{J}\mathcal{I}}\| + \frac{R_{\max}}{\lambda_k^2}\mathrm{osc}(\mathcal{R})\right) \gg \mathcal{E}_k^{\mathrm{orc}} + \frac{\|E\|}{\lambda_k}.$$

This condition does not require the geometric bias to vanish. It may hold, for instance, when $\mathcal{J}$ contains strong, well-separated spikes that can be accurately estimated, even if the variance profile $\mathcal{R}$ has a large coupling between $U_{\mathcal{J}}$ and $U_{\mathcal{I}}$. A systematic analysis of optimal fully data-driven bias correction is left for future work.

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